Output Sliding Mode Control

Consider the following system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) + \boldsymbol{d}(t)$$
$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) \tag{1}$$

and design the control such that the output $y(t)$ is able to trace the desired trajectory $y_d(t)$. It is known that the Marcov parameters are defined as

$$h_k = \boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{b} \tag{2}$$

and we assume the system satisfies the following condition:

$$\begin{cases} h_k = \boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{b} = 0, \quad \text{for } k = 0, 1, 2, ..., m-1 \\ h_m = \boldsymbol{c}^T\boldsymbol{A}^m\boldsymbol{b} \neq 0, \end{cases} \tag{3}$$

Based on the condition, we obtain

$$\dot{y}(t) = \boldsymbol{c}^T\dot{\boldsymbol{x}}(t) = \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{c}^T\boldsymbol{b}u(t) + \boldsymbol{c}^T\boldsymbol{d}(t) = \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{c}^T\boldsymbol{d}(t) \tag{4}$$

$$\ddot{y}(t) = \boldsymbol{c}^T\boldsymbol{A}\dot{\boldsymbol{x}}(t) + \boldsymbol{c}^T\dot{\boldsymbol{d}}(t) = \boldsymbol{c}^T\boldsymbol{A}^2\boldsymbol{x}(t) + \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{d}(t) + \boldsymbol{c}^T\dot{\boldsymbol{d}}(t) \tag{5}$$

$$y^{(3)}(t) = \boldsymbol{c}^T\boldsymbol{A}^3\boldsymbol{x}(t) + \boldsymbol{c}^T\boldsymbol{A}^2\boldsymbol{d}(t) + \boldsymbol{c}^T\boldsymbol{A}\dot{\boldsymbol{d}}(t) + \boldsymbol{c}^T\ddot{\boldsymbol{d}}(t) \tag{6}$$

$$y^{(k)}(t) = \boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{x}(t) + \sum_{i=0}^{k-1}\boldsymbol{c}^T\boldsymbol{A}^i\boldsymbol{d}^{(k-i-1)}(t), \text{ for } k \leq m \tag{7}$$

$$y^{(m+1)}(t) = \boldsymbol{c}^T\boldsymbol{A}^{m+1}\boldsymbol{x}(t) + h_m u(t) + \sum_{i=0}^{m}\boldsymbol{c}^T\boldsymbol{A}^i\boldsymbol{d}^{(m-i)}(t) \tag{8}$$

where

$$y^{(k)}(t) = \frac{d^k}{dt^k}y(t), \qquad k=0,\ 1,\ 2,\ldots\ldots \tag{9}$$

Let the error function be

$$e(t) = y(t) - y_d(t) \tag{10}$$

and choose the sliding function as

$$s(t) = e^{(m)}(t) + \int_0^t v(\tau)d\tau \tag{11}$$

where

$$e^{(m)}(t) = \frac{d^m}{dt^m}e(t) \tag{12}$$

Differentiating (11) yields

$$\dot{s}(t) = e^{(m+1)}(t) + v(t) = y^{(m+1)}(t) - y_d^{(m+1)}(t) + v(t) \tag{13}$$

From (8), we have

$$\dot{s}(t) = \boldsymbol{c}^T\boldsymbol{A}^{m+1}\boldsymbol{x}(t) + h_m u(t) + \sum_{i=0}^{m}\boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{d}^{(m-i)}(t) - y_d^{(m+1)}(t) + v(t) \tag{14}$$

Let the control algorithm be

$$u(t) = h_m^{-1}\left(-\boldsymbol{c}^T\boldsymbol{A}^{m+1}\boldsymbol{x}(t) + y_d^{(m+1)}(t) - v(t) - (\alpha(t)+\sigma)\frac{s(t)}{|s(t)|}\right) \tag{15}$$

where the constant $\sigma>0$ and

$$\alpha(t) = \left|\sum_{k=0}^{m}\boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{d}^{(m-k)}(t)\right|_{max} \tag{16}$$

It is easy to show that the reaching and sliding condition

$$s(t)\dot{s}(t) < -\sigma|s(t)| \tag{17}$$

is satisfied. That means the system will be successfully driven to the sliding mode $s(t)=0$ in a finite time.

We already know that the system behavior in the sliding mode can be governed by the equivalent control $u_{eq}(t)$, which is obtained from

$$\dot{s}(t)\big|_{u(t)=u_{eq}(t)} = 0 \tag{18}$$

Clearly, from (14) the equivalent control $u_{eq}(t)$ is

$$u_{eq}(t) = \frac{1}{h_m}\left[-\boldsymbol{c}^T\boldsymbol{A}^{m+1}\boldsymbol{x}(t) - \sum_{i=0}^{m}\boldsymbol{c}^T\boldsymbol{A}^k\boldsymbol{d}^{(m-i)}(t) + y_d^{(m+1)}(t) - v(t)\right] \tag{19}$$

On the other hand, if we choose

$$v(t) = \beta_m e^{(m)}(t) + \beta_{m-1}e^{(m-1)}(t) + \ldots + \beta_2\ddot{e}(t) + \beta_1\dot{e}(t) + \beta_0 e(t) \tag{20}$$

then from (13) we have

$$e^{(m+1)}(t) + \beta_m e^{(m)}(t) + \beta_{m-1}e^{(m-1)}(t) + \ldots + \beta_1\dot{e}(t) + \beta_0 e(t) = 0 \tag{21}$$

It seems obvious that by suitably select all the coefficients β_i, i=0,1,2,..,m, if the roots of (21) are all in the left half complex plane, then the error $e(t)$ will vanish as $t\to\infty$. This implies the control goal is completted. Unfortunately, the above conclusion is wrong since we still have to consider about the internal stability of the system as the error $e(t)\to 0$.

Let's check the internal stability in the sliding mode. First, apply the equivalent control (19) to (1), we have

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \frac{1}{h_m}\boldsymbol{b}\boldsymbol{c}^T\boldsymbol{A}^{m+1}\boldsymbol{x}(t) + \frac{1}{h_m}\boldsymbol{b}\,y_d^{(m+1)}(t) - \frac{1}{h_m}\boldsymbol{b}\,v(t) + \boldsymbol{g}_1(t) \tag{22}$$

where $\boldsymbol{g}_1(t)$ is composed of the terms related to $\boldsymbol{d}(t)$. For simplicity, let's neglect the disturbance $\boldsymbol{d}(t)$, then from (4) to (8) we can rewrite (20) as

$$v(t)=\beta_m \boldsymbol{c}^T \boldsymbol{A}^m \boldsymbol{x}(t)+\beta_{m-1}\boldsymbol{c}^T \boldsymbol{A}^{m-1}\boldsymbol{x}(t)+\ldots+\beta_1\boldsymbol{c}^T \boldsymbol{A}\boldsymbol{x}(t)+\beta_0\boldsymbol{c}^T \boldsymbol{x}(t) \\ -\left(\beta_m\, y_d^{(m)}(t)+\beta_{m-1}\, y_d^{(m-1)}(t)+\ldots+\beta_1\, \dot{y}_d(t)+\beta_0\, y_d(t)\right) \tag{23}$$

and (22) as

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)-\frac{1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^{m+1}\boldsymbol{x}(t)-\frac{\beta_m}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^m \boldsymbol{x}(t) \\ -\cdots-\frac{\beta_2}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^2 \boldsymbol{x}(t)-\frac{\beta_1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}\boldsymbol{x}(t)-\frac{\beta_0}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{x}(t)+\boldsymbol{g}_2(t) \tag{24}$$

where $\boldsymbol{g}_2(t)$ is composed of the terms related to $y_d(t)$. Since $y_d(t)$ will not affect the stability, we also neglect it. Hence, the system stabilty is determined by

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)-\frac{1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^{m+1}\boldsymbol{x}(t)-\frac{\beta_m}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^m \boldsymbol{x}(t) \\ -\cdots-\frac{\beta_2}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^2 \boldsymbol{x}(t)-\frac{\beta_1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}\boldsymbol{x}(t)-\frac{\beta_0}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{x}(t) \tag{25}$$

whose characteristic equation is

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}+\frac{1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^{m+1}+\frac{\beta_m}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}^m+\cdots+\frac{\beta_1}{h_m}\boldsymbol{b}\boldsymbol{c}^T \boldsymbol{A}+\frac{\beta_0}{h_m}\boldsymbol{b}\boldsymbol{c}^T\right|=0 \tag{26}$$

or

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left|\boldsymbol{I}+\frac{\boldsymbol{b}}{h_m}\boldsymbol{c}^T\left(\boldsymbol{A}^{m+1}+\beta_m\boldsymbol{A}^m+\cdots+\beta_1\boldsymbol{A}+\beta_0\boldsymbol{I}\right)\left(\lambda\boldsymbol{I}-\boldsymbol{A}\right)^{-1}\right|=0 \tag{27}$$

Further change (27) into

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(1+\frac{\boldsymbol{c}^T\left(\boldsymbol{A}^{m+1}+\beta_m\boldsymbol{A}^m+\cdots+\beta_1\boldsymbol{A}+\beta_0\boldsymbol{I}\right)\left(\lambda\boldsymbol{I}-\boldsymbol{A}\right)^{-1}\boldsymbol{b}}{h_m}\right)=0 \tag{28}$$

Since $h_m \neq 0$ and

$$\left(\lambda\boldsymbol{I}-\boldsymbol{A}\right)^{-1}=\lambda^{-1}\left(\boldsymbol{I}-\frac{\boldsymbol{A}}{\lambda}\right)^{-1}=\lambda^{-1}\left(\boldsymbol{I}+\frac{\boldsymbol{A}}{\lambda}+\frac{\boldsymbol{A}^2}{\lambda^2}+\frac{\boldsymbol{A}^3}{\lambda^3}+\cdots\right) \tag{29}$$

we have

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(h_m+\frac{1}{\lambda}\sum_{j=0}^{m+1}\sum_{k=0}^{\infty}\beta_j\,\frac{\boldsymbol{c}^T\boldsymbol{A}^{k+j}\boldsymbol{b}}{\lambda^k}\right)=0 \tag{30}$$

with β_{m+1}=1. It can be further changed as

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(h_m+\frac{1}{\lambda}\sum_{j=0}^{m+1}\sum_{i=j}^{\infty}\beta_j\,\frac{\boldsymbol{c}^T\boldsymbol{A}^{i}\boldsymbol{b}}{\lambda^{i-j}}\right)=0 \tag{31}$$

or

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(h_m+\lambda^m\sum_{i=m+1}^{\infty}\frac{\boldsymbol{c}^T\boldsymbol{A}^i\boldsymbol{b}}{\lambda^i}+\frac{1}{\lambda}\sum_{j=0}^{m}\beta_j\lambda^j\sum_{i=j}^{\infty}\frac{\boldsymbol{c}^T\boldsymbol{A}^i\boldsymbol{b}}{\lambda^i}\right)=0 \qquad (32)$$

From (3), we obtain

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(h_m+\lambda^m\sum_{i=m+1}^{\infty}\frac{h_i}{\lambda^i}+\frac{1}{\lambda}\sum_{j=0}^{m}\beta_j\lambda^j\sum_{i=j}^{\infty}\frac{h_i}{\lambda^i}\right)=0 \qquad (33)$$

or

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(\lambda^m\sum_{i=m}^{\infty}\frac{h_i}{\lambda^i}+\frac{1}{\lambda}\sum_{j=0}^{m}\beta_j\lambda^j\sum_{i=j}^{\infty}\frac{h_i}{\lambda^i}\right)=0 \qquad (34)$$

It is known that the transfer function of the original system is

$$H(s)=\boldsymbol{c}^T(\lambda\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}=\sum_{i=0}^{\infty}\frac{h_i}{\lambda^i}=\sum_{i=m}^{\infty}\frac{h_i}{\lambda^i} \qquad (35)$$

Hence, (34) becomes

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(\lambda^m H(s)+\frac{1}{\lambda}\sum_{j=0}^{m}\beta_j\lambda^j H(s)\right)=0 \qquad (36)$$

i.e.,

$$\frac{1}{\lambda}\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|\left(\lambda^{m+1}+\sum_{j=0}^{m}\beta_j\lambda^j\right)H(s)=0 \qquad (37)$$

Clearly, in the sliding mode the roots of the characteristic equation (26) at least contain the roots of (21)

$$\lambda^{m+1}+\sum_{j=0}^{m}\beta_j\lambda^j=0 \qquad (37)$$

and the zeros of the original system

$$\left|\lambda\boldsymbol{I}-\boldsymbol{A}\right|H(s)=0 \qquad (38)$$

If the system is nonminimum phase, i.e., the zeros of the original system are not all located in the left half complex plane, then the system is unstable.