21. Applications of Fourier Transform to Communication Systems

To show the importance of Fourier transform in the field of communication, we discuss two basic modulation technologies, called the amplitude modelation (AM) and frequency modulation (FM).

A signal to be transmitted in the communication system is generally expressed as the following form:

$$g(t) = f(t) sin(\omega_c t + \phi(t)) \quad (1)$$

where ω_c is the carrier frequency, $f(t)$ is the time-dependent amplitude and $\phi(t)$ is the time-dependent phase. Next, let's study the AM signal, expressed as

$$g(t) = (A + \alpha \cdot v(t)) cos\, \omega_c t \quad (2)$$

where the phase is set to be $\phi(t)=\pi/2$ and the amplitude is given as

$$f(t) = A + \alpha \cdot v(t) \quad (3)$$

with $A>\alpha \cdot v(t)$. Note that the function $cos\, \omega_c t$ in (2) is the carrier and the amplitude $f(t)$ contains the input signal $v(t)$. Usually, it is said that the carrier $cos\, \omega_c t$ is amplitude modulated by the input signal $v(t)$. The basic idea in communication is to design a receiver that yields an output signal $y(t)$ such that $y(t)=v(t)$. However, it is impossible to realize $y(t)=v(t)$ since there always exists noise to distort the transmitted signal.

For simplicity, let's neglect the noise and find the Fourier transform of the transmitted signal $g(t)$ in (2), which is given as

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} (A + \alpha \cdot v(t)) cos\, \omega_c t \cdot e^{-j\omega t} dt \quad (4) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (A + \alpha \cdot v(t))\left(e^{-j(\omega+\omega_c)t} + e^{-j(\omega-\omega_c)t}\right) dt \\ &= \frac{A}{2}\int_{-\infty}^{\infty} \left(e^{-j(\omega+\omega_c)t} + e^{-j(\omega-\omega_c)t}\right) dt \\ &\quad + \frac{\alpha}{2}\int_{-\infty}^{\infty} v(t) e^{-j(\omega+\omega_c)t} dt + \frac{\alpha}{2}\int_{-\infty}^{\infty} v(t) e^{-j(\omega-\omega_c)t} dt \end{aligned}$$

which results in

$$G(\omega) = A\pi\{\delta(\omega+\omega_c) + \delta(\omega-\omega_c)\} + \frac{\alpha}{2}\{V(\omega+\omega_c) + V(\omega-\omega_c)\} \quad (5)$$

Clearly, the transmitted signalconsists of delta functions at $\omega = \pm\omega_c$ and the shifted input signal $v(t)$ from ω to $\omega \pm \omega_c$.

A receiver to get $g(t)$ will recover the input $v(t)$ from it in several ways. Commonly, we can employ an oscillator for demodulation to create the following signal

$$h(t) = g(t) \cdot \cos \omega_c t = \frac{1}{2}(A + \alpha \cdot v(t))(1 + \cos 2\omega_c t) \qquad (6)$$

whose Fourier transform is

$$\begin{aligned} H(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} (A + \alpha \cdot v(t))(1 + \cos 2\omega_c t) \cdot e^{-j\omega t} dt \qquad (7) \\ &= A\pi\{\delta(\omega) + \delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c)\} \\ &\quad + \alpha\left\{V(\omega) + \frac{1}{2}V(\omega + \omega_c) + \frac{1}{2}V(\omega - \omega_c)\right\} \end{aligned}$$

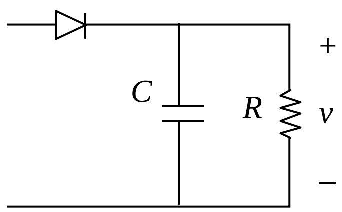
If the signal $h(t)$ passed through an ideal filter given by

$$Q(\omega) = \begin{cases} 1 & -\omega_0 \le \omega \le \omega_0 \\ 0 & otherwise \end{cases} \qquad (8)$$

where $\omega_0 < \omega$. Therefore,the output signal becomes

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)P(\omega)e^{j\omega t} d\omega \qquad (9) \\ &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (\alpha V(\omega) + A\pi\delta(\omega))e^{j\omega t} d\omega \\ &= \alpha v(t) + A\pi \end{aligned}$$

In addition to (6), an alternative demodulation called the asynchronous demodulation is often used to directly detect the envolope of $v(t)$ under the assumption that $v(t)$>0.



Here, we only stress the importance of the Fourier transforms and don’t cover more problems of AM technology in detail. Please refer to other texts.

Now let’s turn to the frequency modulation. The transmitted signal in FM is given by

$$g(t) = \sin(\omega_c t + \phi(t)) \qquad (10)$$

where the amplitude has been normalized to unity and the phase is set to be

$$\phi(t) = \alpha\int_0^t v(\tau)d\tau + \frac{\pi}{2} \qquad (11)$$

Hence,

$$g(t) = cos\left(\omega_c t + \alpha \int_0^t v(\tau) d\tau\right) = cos\,\theta(t) \tag{12}$$

where

$$\theta(t) = \omega_c t + \alpha \int_0^t v(\tau) d\tau \tag{13}$$

The instantaneous frequency then becomes

$$\omega(t) = \omega_c + \alpha v(t) \tag{14}$$

It is obvious that the frequency varies with time due to the input $v(t)$ and the tramsmitted signal $g(t)$ is frequency modulated.

For arbitrary signal $v(t)$, the analysis of FM is rather complex. For simplicity, we assume the input $v(t)$ can be represented by Fourier series and composed of $cos n\omega_0 t$ where ω_0 is the fundamental frequency. Suppose that

$$v(t) = cos\,\omega_0 t \tag{15}$$

then (13) becomes

$$\theta(t) = \omega_c t + k\,sin\,\omega_0 t \tag{16}$$

where $k = \dfrac{\alpha}{\omega_0}$. As a result, we have

$$g(t) = cos(\omega_c t + k\,sin\,\omega_0 t) = cos\,\theta(t) = Re\left(e^{j(\omega_c t + k\,sin\,\omega_0 t)}\right) \tag{17}$$

$$= Re\left(e^{j\omega_c t} e^{jk\,sin\,\omega_0 t}\right)$$

It is known that

$$e^{jk\,sin\,\omega_0 t} = \sum_{n=-\infty}^{\infty} J_n(k) e^{jn\omega_0 t} \tag{18}$$

where J_n is the Bessel function of order n and argument k, shown as below:

k	J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
1.0	0.77	0.44	0.11	0.02	—	—	—	—	—	—
2.0	0.22	0.58	0.35	0.13	0.03	—	—	—	—	—
3.0	−0.26	0.34	0.49	0.31	0.13	0.04	0.01	—	—	—
4.0	−0.40	−0.07	0.36	0.43	0.28	0.13	0.05	0.02	—	—
5.0	−0.18	−0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	—
6.0	0.15	−0.28	−0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02

Therefore,

$$g(t) = Re\left(e^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(k) e^{jn\omega_0 t}\right) \tag{19}$$

whose Fourier transform is

$$G(\omega) = 2\pi \sum_{n=-\infty}^{\infty} J_n(k)\delta(\omega - \omega_c - n\omega_0) \qquad (20)$$

with infinite number of delta functions. From the table, we know that only a few of J_n is important if $k = \frac{\alpha}{\omega_0}$ is small. For example, if k=1 then only the terms for n=0,1,2,3 contribute significantly to $g(t)$.