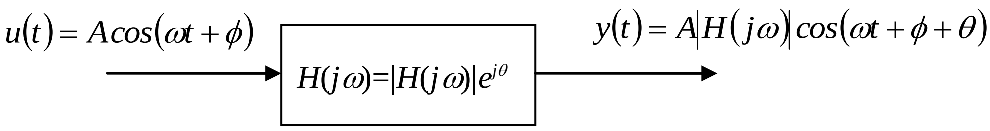
20. Phasor Method for AC Circuits

A circuit driven by sinusoidal current or voltage sources is called an AC circuit. For example, the circuit connected to the electric utility is an AC circuit since the electric utility provides sinusoidal voltage sources. To analyze an AC circuit, we often use the phasor method which is a method derived from the sinusoidal response.

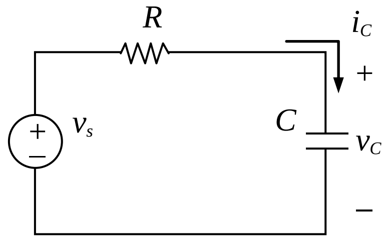
According to the sinusoidal response, when a single-frequency sinusoidal signal $u(t)=A\cos(\omega t+\phi)$ is applied to a system with transfer function $H(s)$, the system output is

$$y(t)=A|H(j\omega)|\cos(\omega t+\phi+\theta) \tag{1}$$

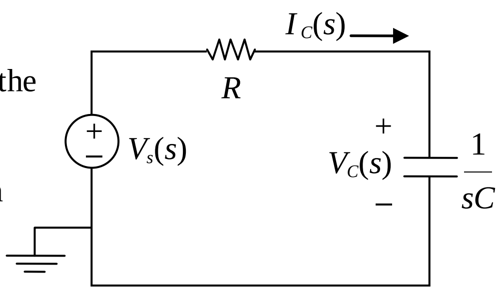
shown as below $H(s)|_{s=j\omega}=|H(j\omega)|e^{j\theta}$.



For example, an RC circuit on the right with R=2$k\Omega$ and C=10μF is excited by a voltage source $v_s(t)=3\cos 25t$ V. Let's determine $v_C(t)$ as $t\to\infty$.



Since the RC circuit always performs stably, we can neglect the initial voltage of the capacitor to solve $v_C(t)$ as $t\to\infty$. Based on Laplace transform, the circuit is redrawn on the right and the capacitor voltage is obtained as



$$V_C(s)=\frac{\frac{1}{sC}}{R+\frac{1}{sC}}V_s(s)=\frac{1}{1+sRC}V_s(s)=H(s)V_s(s) \tag{2}$$

where $H(s)=\frac{1}{1+sRC}$. Since the input $v_s(t)=3\cos 25t$ V, we know that A=3, ω=25 and ϕ=0. Hence, $H(j\omega)\big|_{\omega=25}=\frac{1}{1+j0.5}=\frac{2}{2+j}=\frac{2}{\sqrt{5}}e^{-j26.57^\circ}$, i.e., $|H(j25)|=\frac{2}{\sqrt{5}}$ and

$\theta = -26.57°$. From (1), the voltage $v_C(t)$ is

$$v_C(t) = \frac{6}{\sqrt{5}} cos(25t - 26.57°) = 2.683 cos(25t - 26.57°)$$

or $v_C(t) = 2.4 cos\, 25t + 1.2 sin\, 25t$.

Now, let's introduce the definition of phasor. According to the Euler equation $e^{j\phi} = cos\,\phi + j\, sin\,\phi$, we have $cos\,\phi = Re(e^{j\phi})$ and $sin\,\phi = Im(e^{j\phi})$. This implies the sinusoidal signal $v(t)=V_m cos(\omega t+\phi)$ can be rewritten as

$$v(t) = V_m\, cos(\omega t + \phi) = Re\left(V_m e^{j(\omega t+\phi)}\right) = Re\left(\mathcal{V} e^{j\omega t}\right) \tag{3}$$

where

$$\mathcal{V} = V_m e^{j\phi} = V_m \angle \phi \tag{4}$$

is called the phasor of $v(t)$. Clearly, the phasor includes two parts, the magnitude V_m and the phase ϕ of $v(t)$. For example, $v(t) = -3cos(5t - 20°)$ can be changed into $v(t) = 3cos(5t + 160°)$ whose phasor is $\mathcal{V} = 3\angle 160°$ and $y(t) = 2\,sin(14t + 50°)$ can be rewritten as $y(t) = 2\,cos(14t - 40°)$ whose phasor is $\mathcal{Y} = 2\angle -40°$.

On the other hand, if the frequency is ω=3 rad and the phasors of $v(t)$ and $y(t)$ are $\mathcal{V} = -12 + j5$ and $\mathcal{Y} = j10\, e^{-j15°}$ respectively, i.e., $\mathcal{V} = -12 + j5 = 13\angle 157°$ and $\mathcal{Y} = j10\, e^{-j15°} = 10\angle 75°$, then we can obtain that $v(t) = 13 cos(3t + 157°)$ and $y(t) = 10\, cos(3t + 75°)$.

With the use of phasor, the sinusoidal frequency response can be reformed by setting the input and output as below:

$$u(t) = A cos(\omega t + \phi) = Re\left(Ae^{j(\omega t+\phi)}\right) = Re\left(\mathcal{U} e^{j\omega t}\right) \tag{5}$$

$$\begin{aligned} y(t) &= A|H(j\omega)| cos(\omega t + \phi + \theta) \\ &= Re\left(A|H(j\omega)|e^{j(\omega t+\phi+\theta)}\right) = Re\left(\mathcal{Y} e^{j\omega t}\right) \end{aligned} \tag{6}$$

where their phasors are

$$\mathcal{U} = Ae^{j\phi} = A\angle\phi \tag{7}$$

$$\mathcal{Y} = A|H(j\omega)|e^{j(\phi+\theta)} = A|H(j\omega)|\angle(\phi + \theta) \tag{8}$$

It is obvious that

$$\mathcal{Y} = H(j\omega)\mathcal{U} \tag{9}$$

In other words, the sinusoidal response can be represented by (9) under the condition of single-frequency input.

In addition, the sum of two sinusoidal signals can be also calculated by the sum of their phasors. For example, if $v_1(t) = 12 cos\, 3t$ and $v_2(t) = 4 cos(3t + 30°)$, then their sum is

$$\begin{aligned} v_1(t) + v_2(t) &= 12 cos\, 3t + 4 cos(3t + 30°) \\ &= 12 cos\, 3t + 4(cos\, 3t\, cos\, 30° - sin\, 3t\, sin\, 30°) \\ &= 12 cos\, 3t + 3.464 cos\, 3t - 2 sin\, 3t \\ &= 15.464 cos\, 3t - 2 sin\, 3t \\ &= 15.593 cos(3t + 7.37°) \end{aligned} \tag{10}$$

Now, let's use their phasors $\mathbf{V}_1 = 12\angle 0°$ and $\mathbf{V}_2 = 4\angle 30°$ to calculate their sum as

$$\begin{aligned} \mathbf{V}_1 + \mathbf{V}_2 &= 12\angle 0° + 4\angle 30° = 12 + 3.464 + j2 \\ &= 15.464 + j2 = 15.593\angle 7.37° \end{aligned} \tag{11}$$

which implies $v_1(t) + v_2(t) = 15.593 cos(3t + 7.37°)$, same as the result in (10).

To analyze an AC circuit by the phasor method, the voltages and currents are represented by the phasor such as

$$v(t) = V\, cos(\omega t + \phi_v) = Re\left(Ve^{j(\omega t + \phi_v)}\right) = Re\left(\mathbf{V}e^{j\omega t}\right) \tag{12}$$

$$i(t) = I\, cos(\omega t + \phi_i) = Re\left(Ie^{j(\omega t + \phi_i)}\right) = Re\left(\mathbf{I}e^{j\omega t}\right) \tag{13}$$

where $\mathbf{V} = Ve^{j\phi_v}$ and $\mathbf{I} = Ie^{j\phi_i}$ are the voltage phasor and current phasor. For the components *R*, *L* and *C*, they are transformed to the impedances, a ratio between $\mathbf{V}$ and $\mathbf{I}$, i.e.,

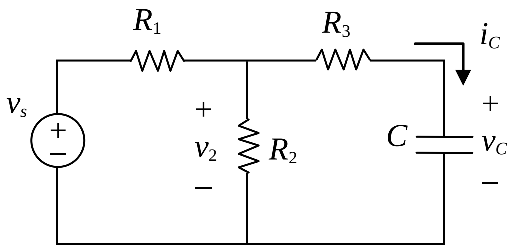
$$\begin{aligned} \mathbf{Z} &= \frac{\mathbf{V}}{\mathbf{I}} = \frac{V}{I} e^{j(\phi_v - \phi_i)} = \frac{V}{I}\angle(\phi_v - \phi_i) \\ &= \frac{V}{I} cos(\phi_v - \phi_i) + j\frac{V}{I} sin(\phi_v - \phi_i) \\ &= R + jX \end{aligned} \tag{14}$$

where $R = \frac{V}{I} cos(\phi_v - \phi_i)$ is the resistance and $X = \frac{V}{I} sin(\phi_v - \phi_i)$ is called the reactance. The units of $\mathbf{Z}$, *R* and *X* are all the same, symboled by Ω.

From the definition given in (14), the impedances of *R*, *L* and *C* can be found as $\mathbf{Z}_R = R$, $\mathbf{Z}_C = \frac{1}{j\omega C}$ and $\mathbf{Z}_L = j\omega L$. Next, let's employ some examples for demonstration.

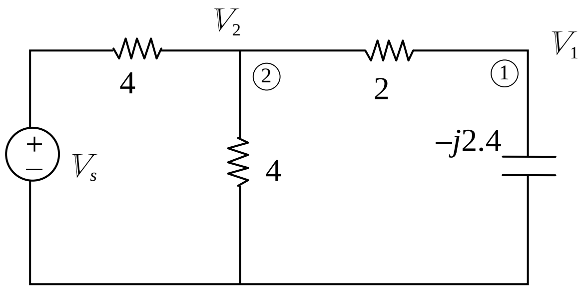
Example

Consider the circuit with R_1=R_2=4 Ω, R_3=2 Ω and C=1/12 F, if $v_s(t)$=4sin5t V, what is $v_2(t)$ as t→∞?



Solve:

Change all the components into impedances as below:



Based on the nodal voltage analysis, we obtain

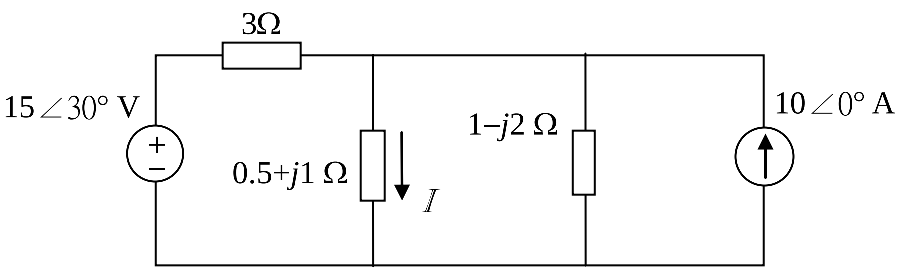
$$\text{KCL①:}\quad \frac{V_1 - V_2}{2} + \frac{V_1}{-j2.4} = 0$$

$$\text{KCL②:}\quad \frac{V_2 - V_s}{4} + \frac{V_2}{4} + \frac{V_2 - V_1}{2} = 0$$

where $V_s = 4\angle(-90°) = -j4$. After calculation, we have $V_2 = 1.339\angle(-109.2°)$, i.e., $v_2(t) = 1.339\,cos(5t - 109.2°) = -0.44\,cos\,5t + 1.26\,sin\,5t$.

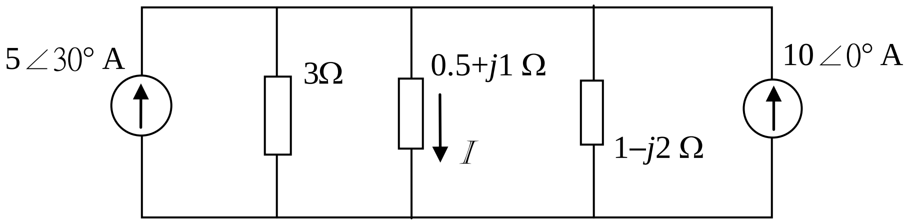
Example

Consider the following AC circuit with frequency ω, then what is the current $i(t)$ whose phasor is I?



Solve

Change the voltage source $15\angle 30°$ V to the current source $5\angle 30°$ A shown as below:



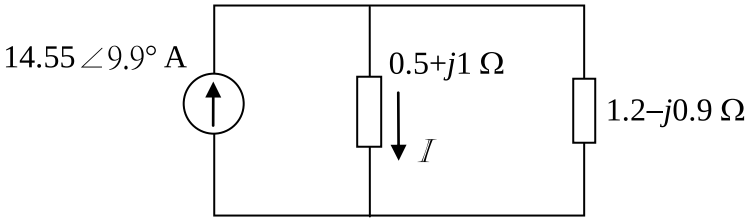
Put the two current sources together and we obtain

$$5\angle 30° + 10\angle 0° = \left(\frac{5\sqrt{3}}{2} + j\frac{5}{2}\right) + 10 = 14.33 + j2.5 = 14.55\angle 9.9°$$

Further, find the parallel impedance of 3Ω and 1–*j*2Ω as

$$\left(\frac{1}{3} + \frac{1}{1 - j2}\right)^{-1} = 1.2 - j0.9$$

then, the circuit is redrawn to be



Hence,

$$\begin{aligned} I &= \frac{1.2 - j0.9}{(0.5 + j1) + (1.2 - j0.9)} \times 14.55\angle 9.9° \\ &= 0.881\angle(-40.24°) \times 14.55\angle 9.9° \\ &= 12.82\angle(-30.34°) \quad \text{(A)} \end{aligned}$$

i.e.,

$$i(t) = 12.82\,cos(\omega t - 30.34°) \quad \text{(A)}$$