

18. Transformations for Signals and Systems

There are six types of transforms which have been introduced in this course, including Fourier series, Fourier transform, Laplace transform, z-transform, discrete-time Fourier transform and discrete Fourier transform. The four types of Fourier transforms are applicable to a variety of problems in signal processing, while the Laplace transform and z-transform are widely used in the design of filters and control systems. Here, we will summarize the properties related to these six transformations before we introduce their applications.

(1) Fourier series for periodic continuous-time signals

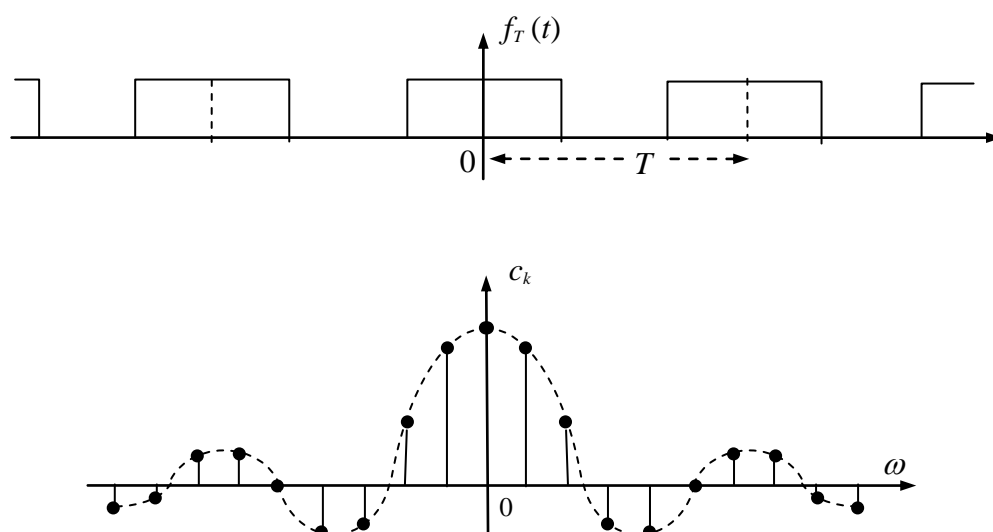
The Fourier series is mainly developed to express a periodic continuous-time signal $f_T(t)$ as the following form

$$f_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \quad (1)$$

where T is the period and $\omega_0 = \frac{2\pi}{T}$. The complex coefficient c_k is treated as the frequency response of $f_T(t)$, expressed as

$$c_k = \frac{1}{T} \int_T f_T(t) e^{-jk\omega_0 t} dt = |c_k| e^{j\phi_k} \quad (2)$$

which is discrete. An example is depicted in the following figures.



Property: With the Fourier series, the periodic continuous-time signal is expressed as (or transformed into) a discrete frequency response.

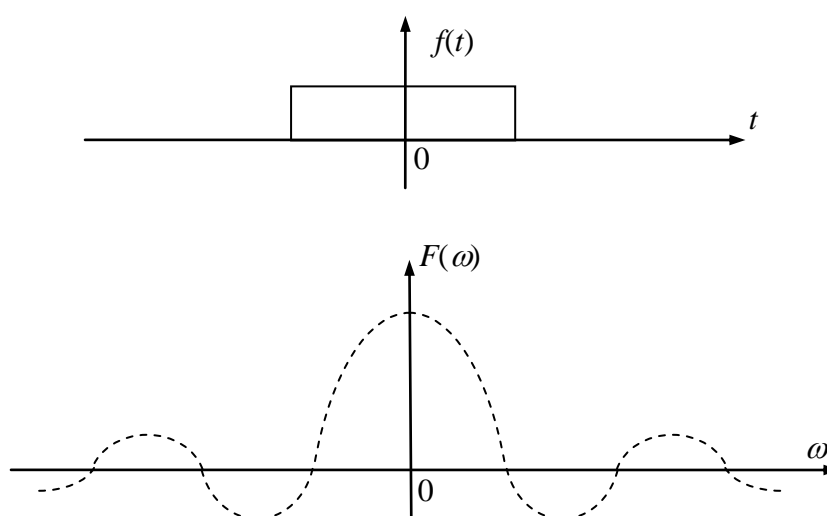
(2) Fourier transform for nonperiodic continuous-time signals

Although the Fourier transform is applicable to a periodic continuous-time signal $f_T(t)$, we will focus on the Fourier transform for nonperiodic continuous-time signals. The pair of Fourier transform and inverse Fourier transform are expressed as below:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt. \quad (3)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega. \quad (4)$$

which can be derived from (1) and (2) as $T \rightarrow \infty$. An example is demonstrated in the following figures.



Property: With the Fourier transform, the nonperiodic continuous-time signal is transformed into continuous-time frequency response.

(3) Laplace transform for LTI continuous-time systems

Let $f(t)$ be a function starting from $t=0$, i.e., $f(t)=0$ for $t<0$, then its Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt \quad (5)$$

and the inverse Laplace transform is given as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st} ds \quad (6)$$

where $s = \sigma + j\omega$ contains the real part $Re(s) = \sigma$ and the imaginary part $Im(s) = \omega$. Clearly, if $s = j\omega$, along the imaginary axis, then (5) becomes

$$F(j\omega) = \int_{0^-}^{\infty} f(t)e^{-j\omega t} dt \quad (7)$$

same as the Fourier transform of $f(t)$. The Laplace transform has been widely applied to LTI continuous systems.

An LTI continuous system is often described by the input-output description or the state-space description. The input-output description is expressed as a differential equation:

$$\begin{aligned} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1\dot{y}(t) + a_0y(t) \\ = b_m u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots + b_1\dot{u}(t) + b_0u(t) \end{aligned} \quad (8)$$

where a_k is constant, $k=0,1,\dots,n$, b_l is constant, $l=0,1,\dots,m$ and $n>m$. The state-space description is given as

$$\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{b} u(t) \quad (9)$$

$$y(t) = \mathbf{c} \cdot \mathbf{x}(t) + du(t) \quad (10)$$

where $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ \cdots \ x_n(t)]^T \in \mathfrak{R}^n$, $\mathbf{A} \in \mathfrak{R}^{n \times n}$, $\mathbf{b} \in \mathfrak{R}^n$ and $\mathbf{c} \in \mathfrak{R}^{1 \times n}$. The system stability is determined by its characteristic equation, shown as

$$|\mathbf{sI} - \mathbf{A}| = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0 \quad (11)$$

If all its roots are located in the complex left half plane, then the system is stable, otherwise it is unstable. Usually, the stability can be tested by the Routh criterion.

(4) z-transform for LTI discrete-time systems

To process discretized systems, we often employ the famous z-transform of a sequence $f[k]$ which is defined as below:

$$\mathcal{Z}\{f[k]\} = F(z) = \sum_{k=0}^{\infty} f[k]z^{-k} \quad (12)$$

where $f[k] = f(kT)$ is the sampled data of $f(t)$ with sampling time T . The inverse z-transform is given by the following integral

$$\mathcal{Z}^{-1}\{F(z)\} = f[k] = \frac{1}{2\pi j} \oint_{\Gamma} F(z)z^{k-1} dz \quad (13)$$

where Γ is the close path of integration.

Similarly, an LTI discrete system is often described by the input-output description or the state-space description. The input-output description is expressed as

a difference equation:

$$y[k+n] + a_{n-1}y[k+n-1] + \dots + a_1y[k+1] + a_0y[k] = b_nu[k+n] + b_{n-1}u[k+n-1] + \dots + b_1u[k+1] + b_0u[k] \quad (14)$$

where n is the order of the discretized system. The state-space description is given as

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k] \quad (15)$$

$$y[k] = \mathbf{c}\mathbf{x}[k] + du[k] \quad (16)$$

where $\mathbf{x}(t) = [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^T \in \mathcal{R}^n$, $\mathbf{A} \in \mathcal{R}^{n \times n}$, $\mathbf{b} \in \mathcal{R}^n$ and $\mathbf{c} \in \mathcal{R}^{1 \times n}$. The system stability is determined by its characteristic equation, shown as

$$|z\mathbf{I} - \mathbf{A}| = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0 = 0 \quad (17)$$

If all its roots are located in the unit circle of the complex plane, then the system is stable, otherwise it is unstable. Usually, the stability can be checked by the Jury's test.

(5) Discrete-time Fourier transform for discrete-time signals

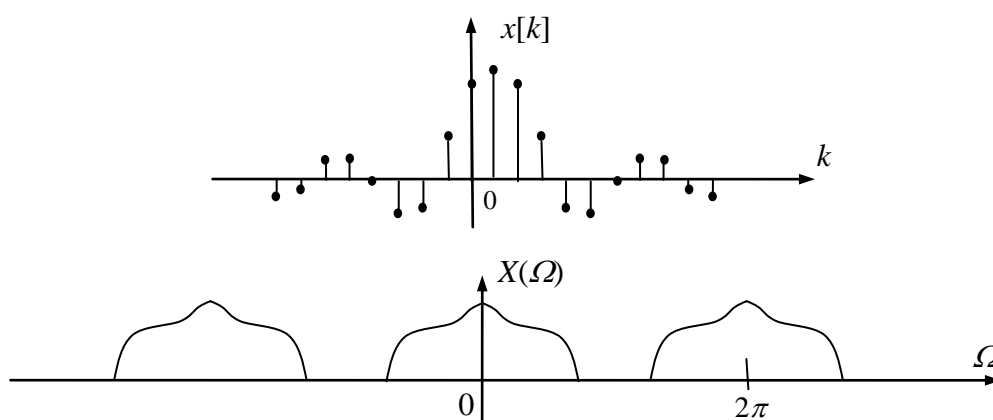
To deal with discrete-time signals, we often adopt the so-called discrete-time Fourier transform, defined as

$$\mathfrak{F}_{DT}\{x[k]\} = X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-jk\Omega} \quad (18)$$

and its inverse discrete-time Fourier transform is

$$\mathfrak{F}_{DT}^{-1}\{X(\Omega)\} = x[k] = \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{jk\Omega} d\Omega \quad (19)$$

where $X(\Omega)$ is periodic with periodicity 2π .



Property: With the discrete-time Fourier transform, the discrete-time signal is transformed into periodic continuous frequency response.

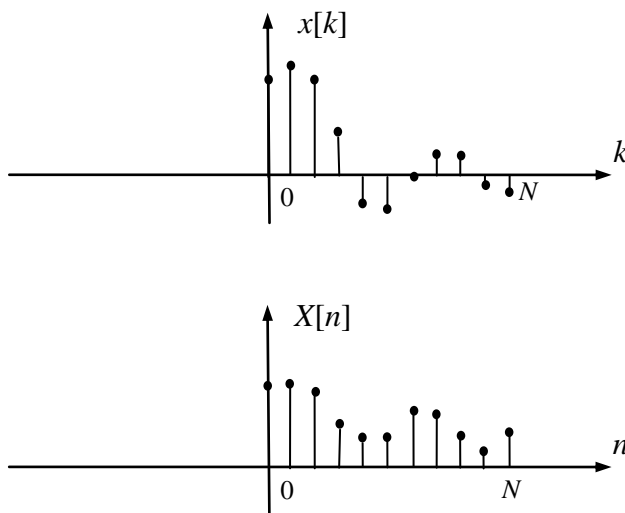
(6) Discrete Fourier transform for discrete-time signals

To make the computation easier, the discrete Fourier transform is developed to transform the discrete-time signal into discrete frequency response. The discrete Fourier transform is defined as

$$\mathfrak{F}_D\{x[k]\} = X[n] = \sum_{k=0}^{N-1} x[k]W_N^{kn}, \quad n=1, 2, \dots, N-1; \quad (20)$$

$$\mathfrak{F}_D^{-1}\{X[n]\} = x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n]W_N^{-kn}, \quad k=1, 2, \dots, N-1. \quad (21)$$

where $W_N = e^{-j\frac{2\pi}{N}}$.



Property: With the discrete Fourier transform, the discrete-time signal is transformed into discrete frequency response.