17. Discrete Fourier Transform

In previous topics, both the Fourier transform and the discrete-time Fourier transform produce continuous functions of frequency. In order to use the digital computer, we need a different transformation which can generate discrete-frequency spectrum for discrete-time samples. One of the transformation is called the discrete Fourier transform or DFT in brief.

It has been introduced that a continuous-time function $x(t)$ can be represented by its sampled values $x[k]=x(kT)$ where T is the sampling time. The pair of DTFT and IDTFT are expreesed as

$$X(\Omega)=\sum_{k=-\infty}^{\infty}x[k]e^{-jk\Omega} \tag{1}$$

$$x[k]=\frac{1}{2\pi}\int_{\Omega_1}^{\Omega_1+2\pi}X(\Omega)e^{jk\Omega}d\Omega \tag{2}$$

where $X(\Omega)$ is a periodic function with period 2π. To use the computer, we have to aplly the approximate signal of $x[k]$, expressed as

$$x_N[k]=\begin{cases}x[k] & 0\le k\le N-1\\ 0 & elsewhere\end{cases} \tag{3}$$

which contains N values of $x[k]$. Besides, the continuous $X(\Omega)$ should be modified by discrete-frequency samples shown as

$$X_N(\Omega)=\sum_{k=-\infty}^{\infty}x_N[k]e^{-jk\Omega}=\sum_{k=0}^{N-1}x[k]e^{-jk\Omega} \tag{4}$$

which is approximated to $X(\Omega)$ if not all the values $x[k]$ are included in $x_N[k]$. Since $X_N(\Omega)$ is still a periodic function with period 2π, we generally select N samples of $X_N(\Omega)$, the same number of samples of $x_N[k]$, to represent the frequency spectrum in one period 2π. Therefore, the sampled frequency spectrum is given by

$$X_S(\Omega)=X_N(\Omega)\sum_{n=0}^{N-1}\delta\left(\Omega-2\pi\frac{n}{N}\right) \tag{5}$$

which results in

$$X_S(\Omega)=\sum_{n=0}^{N-1}X_N\left(2\pi\frac{n}{N}\right)\delta\left(\Omega-2\pi\frac{n}{N}\right) \tag{6}$$

For convenience, we let $X[n]=X_N\left(2\pi\frac{n}{N}\right)$, for n=0,1,2,…,N–1, and from (4) the discrete Fourier transform is defined as

$$\Im_D\{x[k]\} = X[n] = X_N\left(2\pi\frac{n}{N}\right) = \sum_{k=0}^{N-1} x[k] e^{-jkn\frac{2\pi}{N}} \qquad (7)$$

Now, calculate the following summation

$$\sum_{n=0}^{N-1} X[n] e^{jkn\frac{2\pi}{N}} = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m] e^{-jmn\frac{2\pi}{N}}\right) e^{jkn\frac{2\pi}{N}} \qquad (8)$$

$$= \sum_{m=0}^{N-1} x[m] \sum_{n=0}^{N-1} e^{-j(m-k)n\frac{2\pi}{N}}$$

Since

$$\sum_{n=0}^{N-1} e^{-j(m-k)n\frac{2\pi}{N}} = \begin{cases} N & m = k \\ 0 & m \neq k \end{cases} \qquad (9)$$

we rewrite (8) as

$$\sum_{n=0}^{N-1} X[n] e^{jkn\frac{2\pi}{N}} = Nx[k] \qquad (10)$$

Therefore, we have the inverse discrete Fourier Transform, or IDFT in brief, as below:

$$\Im_D^{-1}\{X[n]\} = x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] e^{jkn\frac{2\pi}{N}} \qquad (11)$$

To sum up, the transform pair developed for N discrete-frequency sampled calculated from N discrete-time samples is known as *DFT* and *IDFT*, both respectively expressed by

$$\Im_D\{x[k]\} = X[n] = \sum_{k=0}^{N-1} x[k] e^{-jkn\frac{2\pi}{N}}, \; n=1, 2, \ldots, N-1; \qquad (11)$$

$$\Im_D^{-1}\{X[n]\} = x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] e^{jkn\frac{2\pi}{N}}, \; k=1, 2, \ldots, N-1. \qquad (12)$$

Moreover, we often define $W_N = e^{-j\frac{2\pi}{N}}$ and thus

$$\sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N & k = 0 \\ 0 & k = 1,2,\ldots,N-1 \end{cases} \qquad (13)$$

In addition, (11) and (12) can be rewritten into

$$\Im_D\{x[k]\} = X[n] = \sum_{k=0}^{N-1} x[k] W_N^{kn}, \; n=1, 2, \ldots, N-1; \qquad (14)$$

$$\Im_D^{-1}\{X[n]\} = x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] W_N^{-kn}, \; k=1, 2, \ldots, N-1. \qquad (15)$$

In general, both the DFT and IDFT are respectivelly stated by (14) and (15).

Sinec the DFT $X[n]$ has a resolution of $2\pi/N$ for N samples of $x[k]$, if a DFT of higher resolution is required, we could use the zero padding process to increase the number N by adding zero-valued samples to $x[k]$.

Example Find the DFT for $x[k]=k+1$, $k=0,1,2,3$, for $N=4$ and $N=8$.

Sol:

(A) $N=4$, $W_4 = e^{-j2\pi/4} = -j$, $X[n] = \sum_{k=0}^{3} x[k] W_4^{n\times k}$

$$X[0] = x[0] + x[1] + x[2] + x[3] = 1 + 2 = 3 + 4 = 10$$

$$X[1] = x[0] + x[1](-j) + x[2](-j)^2 + x[3](-j)^3 = 1 - 2j - 3 + 4j = -2 + 2j$$

$$X[2] = x[0] + x[1](-j)^2 + x[2](-j)^4 + x[3](-j)^6 = 1 - 2 + 3 - 4 = -2$$

$$X[3] = x[0] + x[1](-j)^3 + x[2](-j)^6 + x[3](-j)^9 = 1 + 2j - 3 - 4j = -2 - 2j$$

```
This MATLAB program is for DFT of N=4
>> N=4;
>> x=[1 2 3 4];
>> for kk=1:N
            X(kk)=0;
            k=kk-1;
            for nn=1:N;
                   n=nn-1;
                   X(kk)=X(kk)+x(nn)*exp(-j*2*pi*k*n/N);
            end
       end
>> x
x =
        1         2         3         4
>> X
X =
   10.0000    -2.0000 + 2.0000i    -2.0000 - 0.0000i    -2.0000 - 2.0000i
```

(A) $N=8$, $W_8 = e^{-j2\pi/8} = e^{-j\pi/4}$, $X[n] = \sum_{k=0}^{7} x[k] W_8^{n\times k}$

$$X[0] = x[0] + x[1] + x[2] + x[3] = 1 + 2 = 3 + 4 = 10$$

$$X[1] = x[0] + x[1]e^{-j\pi/4} + x[2]e^{-j2\pi/4} + x[3]e^{-j3\pi/4}$$
$$= 1 + \sqrt{2}(1 - j) + 3(-j) + 2\sqrt{2}(-1 - j) = 1 - \sqrt{2} - j\left(3 + 3\sqrt{2}\right)$$

$$X[2] = x[0] + x[1]e^{-j2\pi/4} + x[2]e^{-j4\pi/4} + x[3]e^{-j6\pi/4}$$
$$= 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2$$

$$X[3] = x[0] + x[1]e^{-j3\pi/4} + x[2]e^{-j6\pi/4} + x[3]e^{-j9\pi/4}$$
$$= 1 + \sqrt{2}(-1 - j) + 3(j) + 2\sqrt{2}(1 - j) = 1 + \sqrt{2} + j\left(3 - 3\sqrt{2}\right)$$

$$X[4] = ...$$

This MATLAB program is for DFT of N=8

```
>> N=8;
>> x=[1 2 3 4 0 0 0 0];
>> for kk=1:N
        X(kk)=0;
        k=kk-1;
        for nn=1:N;
            n=nn-1;
            X(kk)=X(kk)+x(nn)*exp(-j*2*pi*k*n/N);
        end
    end
>> x
x =
     1     2     3     4     0     0     0     0
>> X
X =
Columns 1 through 4
  10.0000    -0.4142 - 7.2426i    -2.0000 + 2.0000i    2.4142 - 1.2426i
Columns 5 through 8
  -2.0000 - 0.0000i    2.4142 + 1.2426i    -2.0000 - 2.0000i    -0.4142 + 7.2426i
```

==

The DFT of N-sample $x[k]$ requires N^2 multiplication. To compute DFT more efficiently, we often employ the so-called fast Fourier transform, FFT in brief, which compute DFT with N=2^m samples and approximately requires $N(log_2 N)$ multiplication, much less than N^2. There are three important properties of $W_N = e^{-j\frac{2\pi}{N}}$ for N=2^m, which are

$$W_N^N = e^{-j\frac{2\pi}{N}(N)} = e^{-j2\pi} = 1 \qquad (16)$$

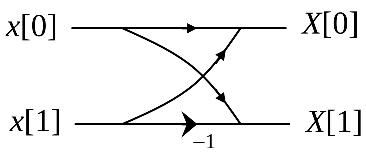
$$W_N^{N/2} = e^{-j\frac{2\pi}{N}\left(\frac{N}{2}\right)} = e^{-j\pi} = -1 \qquad (17)$$

$$W_N^{(N/2)+m} = -W_N^m \qquad (18)$$

Based on the above properties, below show the process of FFT for N=2,4 and 8.

N=2, $W_2 = e^{-j\frac{2\pi}{2}} = -1$

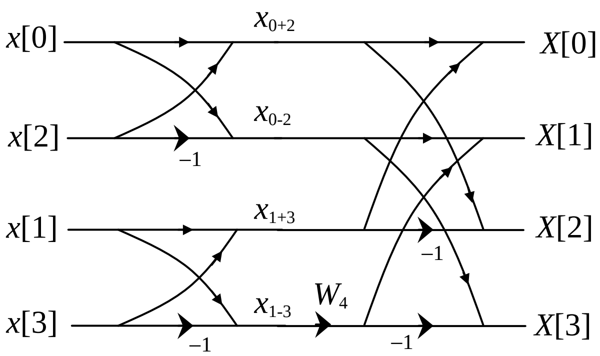
$$\begin{cases} X[0] = x[0]W_2^0 + x[1]W_2^0 = x[0] + x[1] \\ X[1] = x[0]W_2^0 + x[1]W_2^1 = x[0] - x[1] \end{cases}$$



N=4, $W_4 = e^{-j\frac{2\pi}{4}} = -j$

$$\begin{cases} X[0] = x[0]W_4^0 + x[1]W_4^0 + x[2]W_4^0 + x[3]W_4^0 = x[0] + x[1] + x[2] + x[3] \\ X[1] = x[0]W_4^0 + x[1]W_4^1 + x[2]W_4^2 + x[3]W_4^3 = x[0] + x[1]W_4^1 - x[2] - x[3]W_4^1 \\ X[2] = x[0]W_4^0 + x[1]W_4^2 + x[2]W_4^4 + x[3]W_4^6 = x[0] - x[1] + x[2] - x[3] \\ X[3] = x[0]W_4^0 + x[1]W_4^3 + x[2]W_4^6 + x[3]W_4^9 = x[0] - x[1]W_4^1 - x[2] + x[3]W_4^1 \end{cases}$$

$$\begin{cases} X[0] = (x[0] + x[2]) + (x[1] + x[3]) = x_{0+2} + x_{1+3} \\ X[1] = (x[0] - x[2]) + (x[1] - x[3])W_4^1 = x_{0-2} + x_{1-3}W_4 \\ X[2] = (x[0] + x[2]) - (x[1] + x[3]) = x_{0+2} - x_{1+3} \\ X[3] = (x[0] - x[2]) - (x[1] - x[3])W_4^1 = x_{0-2} - x_{1-3}W_4 \end{cases}$$



N=8, $W_8 = e^{-j\frac{2\pi}{8}}$

$$\begin{cases} X[0] = x[0]W_8^0 + x[1]W_8^0 + x[2]W_8^0 + x[3]W_8^0 + x[4]W_8^0 + x[5]W_8^0 + x[6]W_8^0 + x[7]W_8^0 \\ X[1] = x[0]W_8^0 + x[1]W_8^1 + x[2]W_8^2 + x[3]W_8^3 + x[4]W_8^4 + x[5]W_8^5 + x[6]W_8^6 + x[7]W_8^7 \\ X[2] = x[0]W_8^0 + x[1]W_8^2 + x[2]W_8^4 + x[3]W_8^6 + x[4]W_8^8 + x[5]W_8^{10} + x[6]W_8^{12} + x[7]W_8^{14} \\ X[3] = x[0]W_8^0 + x[1]W_8^3 + x[2]W_8^6 + x[3]W_8^9 + x[4]W_8^{12} + x[5]W_8^{15} + x[6]W_8^{18} + x[7]W_8^{21} \\ X[4] = x[0]W_8^0 + x[1]W_8^4 + x[2]W_8^8 + x[3]W_8^{12} + x[4]W_8^{16} + x[5]W_8^{20} + x[6]W_8^{24} + x[7]W_8^{28} \\ X[5] = x[0]W_8^0 + x[1]W_8^5 + x[2]W_8^{10} + x[3]W_8^{15} + x[4]W_8^{20} + x[5]W_8^{25} + x[6]W_8^{30} + x[7]W_8^{35} \\ X[6] = x[0]W_8^0 + x[1]W_8^6 + x[2]W_8^{12} + x[3]W_8^{18} + x[4]W_8^{24} + x[5]W_8^{30} + x[6]W_8^{36} + x[7]W_8^{42} \\ X[7] = x[0]W_8^0 + x[1]W_8^7 + x[2]W_8^{14} + x[3]W_8^{21} + x[4]W_8^{28} + x[5]W_8^{35} + x[6]W_8^{42} + x[7]W_8^{49} \end{cases}$$

$$\begin{cases} X[0] = x[0] + x[1] + x[2] + x[3] + x[4] + x[5] + x[6] + x[7] \\ X[1] = x[0] + x[1]W_8^1 + x[2]W_8^2 + x[3]W_8^3 - x[4] - x[5]W_8^1 - x[6]W_8^2 - x[7]W_8^3 \\ X[2] = x[0] + x[1]W_8^2 - x[2] - x[3]W_8^2 + x[4] + x[5]W_8^2 - x[6] - x[7]W_8^2 \\ X[3] = x[0] + x[1]W_8^3 - x[2]W_8^2 + x[3]W_8^1 - x[4] - x[5]W_8^3 + x[6]W_8^2 - x[7]W_8^1 \\ X[4] = x[0] - x[1] + x[2] - x[3] + x[4] - x[5] + x[6] - x[7] \\ X[5] = x[0] - x[1]W_8^1 + x[2]W_8^2 - x[3]W_8^3 - x[4] + x[5]W_8^1 - x[6]W_8^2 + x[7]W_8^3 \\ X[6] = x[0] - x[1]W_8^2 - x[2] + x[3]W_8^2 + x[4] - x[5]W_8^2 - x[6] + x[7]W_8^2 \\ X[7] = x[0] - x[1]W_8^3 - x[2]W_8^2 - x[3]W_8^1 - x[4] + x[5]W_8^3 + x[6]W_8^2 + x[7]W_8^1 \end{cases}$$

$$
\begin{cases}
X[0] = (x[0]+x[4]) + (x[1]+x[5]) + (x[2]+x[6]) + (x[3]+x[7]) \\
X[1] = (x[0]-x[4]) + (x[1]-x[5])W_8^1 + (x[2]-x[6])W_8^2 + (x[3]-x[7])W_8^3 \\
X[2] = (x[0]+x[4]) + (x[1]+x[5])W_8^2 - (x[2]+x[6]) - (x[3]+x[7])W_8^2 \\
X[3] = (x[0]-x[4]) + (x[1]-x[5])W_8^3 - (x[2]-x[6])W_8^2 - (x[3]-x[7])W_8^5 \\
X[4] = (x[0]+x[4]) - (x[1]+x[5]) + (x[2]+x[6]) - (x[3]+x[7]) \\
X[5] = (x[0]-x[4]) - (x[1]-x[5])W_8^1 + (x[2]-x[6])W_8^2 - (x[3]-x[7])W_8^3 \\
X[6] = (x[0]+x[4]) - (x[1]+x[5])W_8^2 - (x[2]+x[6]) + (x[3]+x[7])W_8^2 \\
X[7] = (x[0]-x[4]) - (x[1]-x[5])W_8^3 - (x[2]-x[6])W_8^2 + (x[3]-x[7])W_8^5
\end{cases}
$$

$$
\begin{cases}
X[0] = x_{0+4} + x_{1+5} + x_{2+6} + x_{3+7} = x_{(0+4)+(2+6)} + x_{(1+5)+(3+7)} \\
X[1] = x_{0-4} + x_{1-5}W_8^1 + x_{2-6}W_8^2 + x_{3-7}W_8^3 = \left(x_{0-4} + x_{2-6}W_8^2\right) + \left(x_{1-5} + x_{3-7}W_8^2\right)W_8 \\
X[2] = x_{0+4} + x_{1+5}W_8^2 - x_{2+6} - x_{3+7}W_8^2 = x_{(0+4)-(2+6)} + x_{(1+5)-(3+7)}W_8^2 \\
X[3] = x_{0-4} + x_{1-5}W_8^3 - x_{2-6}W_8^2 + x_{3-7}W_8^5 = \left(x_{0-4} - x_{2-6}W_8^2\right) + \left(x_{1-5} - x_{3-7}W_8^2\right)W_8^3 \\
X[4] = x_{0+4} - x_{1+5} + x_{2+6} - x_{3+7} = x_{(0+4)+(2+6)} - x_{(1+5)+(3+7)} \\
X[5] = x_{0-4} - x_{1-5}W_8^1 + x_{2-6}W_8^2 - x_{3-7}W_8^3 = \left(x_{0-4} + x_{2-6}W_8^2\right) - \left(x_{1-5} + x_{3-7}W_8^2\right)W_8 \\
X[6] = x_{0+4} - x_{1+5}W_8^2 - x_{2+6} + x_{3+7}W_8^2 = x_{(0+4)-(2+6)} - x_{(1+5)-(3+7)}W_8^2 \\
X[7] = x_{0-4} - x_{1-5}W_8^3 - x_{2-6}W_8^2 - x_{3-7}W_8^5 = \left(x_{0-4} - x_{2-6}W_8^2\right) - \left(x_{1-5} - x_{3-7}W_8^2\right)W_8^3
\end{cases}
$$