16. Discrete-Time Fourier Transform

In sampling process, the continuous signal $x(t)$ is transformed to the discrete-time signal $x_s(t)$ by the use of periodic impulse train $\delta_T(t)$ with period T and frequency $\omega_0 = \frac{2\pi}{T}$. The resulted discrete-time signal is expressed as

$$x_s(t) = x(t)\delta_T(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty}x(t)\delta(t-kT) \tag{1}$$

and taking Fourier transform yields

$$X_s(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X(\omega - k\omega_0) \tag{2}$$

It is known that the recovery of band-limited $x(t)$ from $x_s(t)$ can be explained by Shannon's sampling theorem. In addition to sampling process, the discrete-time signal $x_s(t)$ can be also used to define the discrete-time Fourier transform which will be discussed in this topic.

From (1) and the truth of $x(t)\delta(t-kT) = x(kT)\delta(t-kT)$, the sampled signal can be rewritten as

$$x_s(t) = \sum_{k=-\infty}^{\infty}x(kT)\delta(t-kT) \tag{3}$$

whose Fourier transform is expressed as

$$X_s(\omega) = \sum_{k=-\infty}^{\infty}x(kT)e^{-jk\omega T} \tag{4}$$

Let $x[k] = x(kT)$ and $\omega T = \Omega$ and then (4) becomes

$$X(\Omega) = X_s\left(\frac{\Omega}{T}\right) = \sum_{k=-\infty}^{\infty}x[k]e^{-jk\Omega} \tag{5}$$

where $X(\Omega)$ is the so-called discrete-time Fourier transform or DTFT in short. Here, we also express it as

$$\Im_{DT}\{x[k]\} = X(\Omega) = \sum_{k=-\infty}^{\infty}x[k]e^{-jk\Omega} \tag{6}$$

Further, take the following integration

$$\int_{\Omega_1}^{\Omega_1+2\pi}X(\Omega)e^{jn\Omega}d\Omega = \int_{\Omega_1}^{\Omega_1+2\pi}\sum_{k=-\infty}^{\infty}x[k]e^{-jk\Omega}e^{jn\Omega}d\Omega \tag{7}$$

$$= \sum_{k=-\infty}^{\infty}x[k]\int_{\Omega_1}^{\Omega_1+2\pi}e^{j(n-k)\Omega}d\Omega$$

which leads to

$$\int_{\Omega_1}^{\Omega_1+2\pi} X(\Omega)e^{jn\Omega}d\Omega = 2\pi\cdot x[n] \tag{8}$$

since

$$\int_{\Omega_1}^{\Omega_1+2\pi} e^{j(n-k)\Omega}d\Omega = \begin{cases} 2\pi & k=n \\ 0 & k\neq n \end{cases} \tag{9}$$

From (8), it is obvious that

$$\Im_{DT}^{-1}\{X(\Omega)\} = x[k] = \frac{1}{2\pi}\int_{\Omega_1}^{\Omega_1+2\pi} X(\Omega)e^{jk\Omega}d\Omega \tag{10}$$

which is the inverse discrete-time Fourier transform, or IDTFT in short.

Now, let's compare (6) to the z-transform $\mathcal{Z}\{x[k]\} = \sum_{k=0}^{\infty} x[k]\cdot z^{-k}$, we can find that

$$\Im_{DT}\{x[k]\} = X(\Omega) = \sum_{k=0}^{\infty} x[k]e^{-jk\Omega} = \sum_{k=0\infty}^{\infty} x[k]\cdot z^{-k}\Big|_{z=e^{j\Omega}} \tag{11}$$

In other words, we can use all the properties of z-transform satifying $z = e^{j\Omega}$ for the DTFT of $x[k]$ for $k\geq 0$.

For example, the z-transform of a^k for $k\geq 0$ is $\frac{z}{z-a}$ where $|z|>|a|$. By the use of $z=e^{j\Omega}$, we can obtain

$$\Im_{DT}\{a^k\} = \frac{z}{z-a}\Big|_{z=e^{j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}-a} = \frac{1}{1-ae^{-j\Omega}}, \qquad |a|<1 \tag{12}$$

Note that (12) is only true for $|a|<1$ since $|z| = |e^{j\Omega}| = 1$. That implies for $k\geq 0$ we have $\Im_{DT}\{0.5^k\} = \frac{1}{1-0.5e^{-j\Omega}}$ and $\Im_{DT}\{2^k\}$ does not exist.

Some important properties related to the discrete-time Fourier transform will be further discussed below:

(1) Periodicity

$$X(\Omega+2\pi) = X(\Omega) \tag{13}$$

(2) Linearity

$$\Im_{DT}\{ax[k]+by[k]\} = a\Im_{DT}\{x[k]\} + b\Im_{DT}\{y[k]\} \tag{14}$$

(3) Time-shifting

Consider an m-delayed signal $x[k–m]$, whose DTFT is

$$\Im_{DT}\{x[k-m]\} = \sum_{k=0}^{\infty} x[k-m]e^{-jk\Omega} = \sum_{p=-m}^{\infty} x[p]e^{-j(p+m)\Omega} \tag{15}$$

$$= \sum_{p=0}^{\infty} x[p]e^{-j(p+m)\Omega} = e^{-jm\Omega}\left(\sum_{p=0}^{\infty} x[p]e^{-jp\Omega}\right) = e^{-jm\Omega}X(\Omega)$$

(4) Premultiplying $e^{-jk\Omega_0}$ to $x[k]$

$$\Im_{DT}\{e^{-jk\Omega_0}x[k]\} = \sum_{k=0}^{\infty} e^{-jk\Omega_0}x[k]e^{-jk\Omega} = \sum_{k=0}^{\infty} x[k]e^{-jk(\Omega-\Omega_0)} \tag{16}$$

$$= X(\Omega-\Omega_0)$$

(5) Real $x[k]$

From the DTFT, we have

$$X(\Omega) = Re[X(\Omega)] + j\,Im[X(\Omega)] \tag{17}$$

where

$Re[X(\Omega)]$ is even; $Im[X(\Omega)]$ is odd;

$|X(\Omega)|$ is even; $\angle X(\Omega)$ is odd.

(6) Convolution in time

Define the convolution of $x[k]$ and $y[k]$ as below:

$$x[k]*y[k] = \sum_{m=0}^{\infty} x[k-m]y[m] \tag{18}$$

The DTFT of $x[k]*y[k]$ is then obtained as

$$\Im_{DT}\{x[k]*y[k]\} = X(\Omega)Y(\Omega) \tag{19}$$

(7) Convolution in frequency

The DTFT of $x[k]y[k]$ is obtained as

$$\Im_{DT}\{x[k]y[k]\} = \frac{1}{2\pi}X(\Omega)*Y(\Omega) \tag{20}$$

(8) Parseval's Theorem

The Parseval's theorem for discrete signals is given by

$$\sum_{k=-\infty}^{\infty}|x[k]|^2 = \frac{1}{2\pi}\int_{2\pi}|X(\Omega)|^2 d\Omega \tag{21}$$

Next, before we introduce the discrete Fourier transform, or DFT in short, Let's discuss the DTFT of periodic signals.

Consider a periodic signal $x[k]$ with period N where N is an integer and a finite duration signal $x_0[k]$ which is given as

$$x_0[k]=\begin{cases} x[k], & 0\le k\le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

The DTFT of $x_0[k]$ is obtained as

$$\Im_{DT}\{x_0[k]\}=X_0(\Omega)=\sum_{k=0}^{N-1}x[k]e^{-jk\Omega} \tag{23}$$

Then, the periodic signal can be expressed as

$$x[k]=x_0[k]*\sum_{n=-\infty}^{\infty}\delta[k-nN] \tag{24}$$

whose DTFT is

$$X(\Omega)=X_0(\Omega)\sum_{n=-\infty}^{\infty}\Im_{DT}\{\delta[k-nN]\} \tag{24}$$

$$=X_0(\Omega)\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta[k-nN]e^{-jk\Omega}=X_0(\Omega)\sum_{n=-\infty}^{\infty}e^{-jnN\Omega}$$

From the Fourier series of impulse train, we know that

$$\delta_T(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\frac{2\pi}{T}t}=\sum_{n=-\infty}^{\infty}\frac{1}{T}e^{-jn\frac{2\pi}{T}t} \tag{25}$$

which implies

$$\frac{N}{2\pi}\sum_{n=-\infty}^{\infty}e^{-jnN\Omega}=\sum_{n=-\infty}^{\infty}\delta\left(\Omega-n\frac{2\pi}{N}\right) \tag{26}$$

by changing t and T into Ω and $2\pi/N$ respectively. Therefore, we have

$$X(\Omega)=X_0(\Omega)\sum_{n=-\infty}^{\infty}e^{-jnN\Omega}=\frac{2\pi}{N}X_0(\Omega)\sum_{n=-\infty}^{\infty}\delta\left(\Omega-n\frac{2\pi}{N}\right) \tag{27}$$

$$=\frac{2\pi}{N}\sum_{n=-\infty}^{\infty}X_0(\Omega)\delta\left(\Omega-n\frac{2\pi}{N}\right)$$

$$=\frac{2\pi}{N}\sum_{n=-\infty}^{\infty}X_0\left(n\frac{2\pi}{N}\right)\delta\left(\Omega-n\frac{2\pi}{N}\right)$$

Since $X(\Omega)$ is periodic with period 2π, the N distinct $x[k]$ for $0\le k\le N-1$ is transformed into N distinct $X_0(2n\pi/N)$ for $0\le n\le N-1$. The IDTFT is then obtained as

$$\Im_{DT}^{-1}\{X(\Omega)\} = x[k] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega) e^{jk\Omega} d\Omega \tag{28}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} X_0\left(n\frac{2\pi}{N}\right)\int_0^{2\tau} \delta\left(\Omega - n\frac{2\pi}{N}\right) e^{jk\Omega} d\Omega$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} X_0\left(n\frac{2\pi}{N}\right) e^{jkn\frac{2\pi}{N}}$$

where

$$X_0\left(n\frac{2\pi}{N}\right) = \sum_{k=0}^{N-1} x[k] e^{-jkn\frac{2\pi}{N}} \tag{29}$$

Clearly, there are N distinct values of $x[k]$ and N distinct values of $X_0(2n\pi/N)$. This is a very important observation. With these discrete values, we can easily calculate them in a digital way, called discrete Fourier transform or DFT in short, unlike the DTFT $X(\Omega)$ which is continuous in frequency Ω.