15. LTI System in Difference Equation

In system engineering, an *n*th-order discretized LTI system can be described by the following advanced form:

$$y[k+n]+a_{n-1}y[k+n-1]+\cdots+a_1y[k+1]+a_0y[k] \tag{1}$$
$$=b_nu[k+n]+b_{n-1}u[k+n-1]+\cdots+b_1u[k+1]+b_0u[k]$$

which in fact is equivalent to the following delayed form

$$y[k]+a_{n-1}y[k-1]+\cdots+a_1y[k-n+1]+a_0y[k-n] \tag{2}$$
$$=b_nu[k]+b_{n-1}u[k-1]+\cdots+b_1u[k-n+1]+b_0u[k-n]$$

In this course we will focus on the delayed form.

Assume $h[k]$ is the impulse response of a causal system and then $h[k]=0$ for $k<0$. It is known that for an initially relaxed system, i.e., $y[k]=u[k]=0$ for $k<0$, the input and output can be described by the following convolution

$$y[k]=h[k]*u[k]=\sum_{m=0}^{k}h[k-m]u[m] \tag{3}$$

which leads to

$$Y(z)=H(z)U(z) \tag{4}$$

where $H(z)$ is the transfer function of the system. According to (2), taking z-transform yields

$$Y(z)+a_{n-1}z^{-1}Y(z)+\cdots+a_1z^{-(n-1)}Y(z)+a_0z^{-n}Y(z) \tag{5}$$
$$=b_nU(z)+b_{n-1}z^{-1}U(z)+\cdots+b_1z^{-(n-1)}U(z)+b_0z^{-n}U(z)$$

and thus

$$Y(z)=\frac{b_n+b_{n-1}z^{-1}+\cdots+b_1z^{-(n-1)}+b_0z^{-n}}{1+a_{n-1}z^{-1}+\cdots+a_1z^{-(n-1)}+a_0z^{-n}}U(z) \tag{6}$$

Compared to (4), it is clear that the transfer function is

$$H(z)=\frac{Y(z)}{U(z)}=\frac{b_n+b_{n-1}z^{-1}+\cdots+b_1z^{-(n-1)}+b_0z^{-n}}{1+a_{n-1}z^{-1}+\cdots+a_1z^{-(n-1)}+a_0z^{-n}} \tag{7}$$

which is in negative-power form. We can rewrite (7) into a positive-power form as

$$H(z)=\frac{b_nz^n+b_{n-1}z^{n-1}+\cdots+b_1z+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0} \tag{8}$$

which is the form will be used in this course.

Similar to the Laplace transform, the discrete signals given as z-transform is also solved by using partial fraction expansion and then taking inverse z-transform. Here, we will adopt some examples for demonstration.

Example

What is the discrete signal $y[k]$ for $k \geq 0$ whose z-transform is

$$Y(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Sol:

First, premultiply $Y(z)$ by z^2 and turn it into the positive-power form as

$$Y(z) = \frac{z^2}{(z-1)(z-0.5)}$$

Further express it as below:

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

Hence, we have

$$z = A_1(z-0.5) + A_2(z-1)$$

and obtain $A_1 = 2$ and $A_1 = -1$. That results in

$$Y(z) = 2\frac{z}{z-1} - 1\frac{z}{z-0.5}$$

Taking the inverse z-transform, the discrete signal $y[k]$ is

$$y[k] = 2 - (0.5)^k \qquad \text{for } k \geq 0.$$

Example

An initially relaxed LTI discrete system is given as

$$y[k] - y[k-1] + 0.5y[k-2] = u[k] + u[k-1]$$

(A) What is the transfer function? Is the system stable?

(B) What is the impulse response?

(C) If $u[k]$ is a unit step input, then what is the output response $y[k]$?

Sol:

(A)

Taking the z-transform yields $Y(z)-z^{-1}Y(z)+0.5z^{-2}Y(z)=U(z)+z^{-1}U(z)$, and then we obtain the transfer function as

$$H(z)=\frac{Y(z)}{U(z)}=\frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}=\frac{z^2+z}{z^2-z+0.5}$$
$$=\frac{z+1}{\left(z-0.707e^{-j45^\circ}\right)\left(z-0.707e^{j45^\circ}\right)}$$

where the two poles are located in the unit circle and then the system is stable.

(B)

Further express $H(z)$ as below:

$$\frac{H(z)}{z}=\frac{z+1}{z^2-z+0.5}=\frac{z+1}{(z-0.5+j0.5)(z-0.5-j0.5)}$$
$$=\frac{A}{z-0.707e^{-j45^\circ}}+\frac{A^*}{z-0.707e^{j45^\circ}}$$

Hence, we have

$$z+1=A(z-0.5-j0.5)+A^*(z-0.5+j0.5)$$

and obtain $A=0.5+j1.5=\frac{\sqrt{10}}{2}e^{j71.6^\circ}$. That results in

$$H(z)=\frac{\sqrt{10}}{2}e^{j71.6^\circ}\frac{z}{z-0.707e^{-j45^\circ}}+\frac{\sqrt{10}}{2}e^{-j71.6^\circ}\frac{z}{z-0.707e^{j45^\circ}}$$

Taking the inverse z-transform, the impulse response $h[k]$ is

$$h[k]=\frac{\sqrt{10}}{2}e^{j71.6^\circ}\left(0.707e^{-j45^\circ}\right)^k+\frac{\sqrt{10}}{2}e^{-j71.6^\circ}\left(0.707e^{j45^\circ}\right)^k$$
$$=\frac{\sqrt{10}}{2}(0.707)^k e^{-j(45^\circ k-71.6^\circ)}+\frac{\sqrt{10}}{2}(0.707)^k e^{j(45^\circ k-71.6^\circ)}$$
$$=\sqrt{10}(0.707)^k\cos(45^\circ k-71.6^\circ)$$

(C)

Since $U(z)=\frac{z}{z-1}$, we have

$$Y(z)=H(z)U(z)=\frac{z(z^2+z)}{(z-1)(z^2-z+0.5)}$$
$$=\frac{A_1z}{z-1}+\frac{A_2z}{z-0.707e^{-j45^\circ}}+\frac{A_2^*z}{z-0.707e^{j45^\circ}}$$

where $A_1 = 4$ and $A_2 = 1.58e^{j161.6^\circ}$. Hence, we have

$$y[k] = 4 + 3.16(0.707)^k \cos(45^\circ k - 161.6^\circ)$$

Exercise:

An initially relaxed LTI discrete system is given as

$$y[k] + 0.1y[k-1] + 0.2y[k-2] = u[k] + u[k-1]$$

(A) What is the transfer function? Is the system stable?

(B) What is the impulse response?

(C) If $u[k]$ is a unit step input, then what is the output response $y[k]$?