14. Discretized Systems and z-Transform

Recently, signals in digital form have been widely used, such as voice, music and photo. To deal with these kinds of digital signals, it is required to use digital systems. In this course we will focus on the discrete signals, which are different tfrom the digital signals in quantization.

It is easy to understand that the discrete signals can be obtained by sampling. Usually, the discrete signals are denoted as $x[k]$, k=0,1,2,3,…. Actually, in some cases we can also achieve discrete signals from a continuous system by sending stepwise input such as $u(t)=u(kT)$ for $kT\le t<(k+1)T$, k=0,1,2,…. This is related to a technique called system discretization which will be discussed below.

In continuous systems, it is known that an n-th order system can be detailedly represented by the state-space description given as below:

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\cdot\boldsymbol{x}(t)+\boldsymbol{b}u(t),\qquad \boldsymbol{x}(0)=\boldsymbol{x}_0 \tag{1}$$

$$y(t)=\boldsymbol{c}\cdot\boldsymbol{x}(t)+du(t) \tag{2}$$

where $\boldsymbol{x}(t)\in\Re^n$. Then, the system state is attained as

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\,\boldsymbol{x}_0+\int_0^t e^{\boldsymbol{A}(t-\tau)}\,\boldsymbol{b}u(\tau)d\tau \tag{3}$$

If the input $u(t)$ is in stepwise, i.e., $u(t)=u(kT)$ for $kT\le t<(k+1)T$, k=0,1,2,…, where T is often related to the sampling time, then

$$\boldsymbol{x}(kT)=e^{kT\boldsymbol{A}}\boldsymbol{x}_0+\int_0^{kT}e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(\tau)d\tau \tag{4}$$

As a result, we have

$$\begin{aligned}\boldsymbol{x}((k+1)T)&=e^{(k+1)T\boldsymbol{A}}\boldsymbol{x}_0+\int_0^{(k+1)T}e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}u(\tau)d\tau \\ &=e^{T\boldsymbol{A}}\left(e^{kT\boldsymbol{A}}\boldsymbol{x}_0+\int_0^{(k+1)T}e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(\tau)d\tau\right) \\ &=e^{T\boldsymbol{A}}\left(\boldsymbol{x}(kT)+\int_{kT}^{(k+1)T}e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(kT)d\tau\right) \\ &=e^{T\boldsymbol{A}}\boldsymbol{x}(kT)+\left(\int_{kT}^{(k+1)T}e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}d\tau\right)u(kT)\end{aligned} \tag{5}$$

Define $\boldsymbol{x}[k]=\boldsymbol{x}(kT)$ and $u[k]=u(kT)$, then (5) can be rearranged as

$$\boldsymbol{x}[k+1]=\boldsymbol{A}_T\boldsymbol{x}[k]+\boldsymbol{b}_Tu[k] \tag{6}$$

where $\boldsymbol{A}_T=e^{T\boldsymbol{A}}$ and $\boldsymbol{b}_T=\int_{kT}^{(k+1)T}e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}d\tau$. Without loss of generality, we rewrite

(6) as below:

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \tag{7}$$

and then the output in (4) becomes

$$y[k] = \boldsymbol{c}\boldsymbol{x}[k] + du[k] \tag{8}$$

According to Cayley-Hamilton theorem, we have

$$\begin{aligned} & y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] \\ & = b_n u[k+n] + b_{n-1}u[k+n-1] + \cdots + b_1 u[k+1] + b_0 u[k] \end{aligned} \tag{9}$$

where n is the order of the discretized system, same as the order of the continuous system. One thing we have to notice: Although the sampling time T does not exist in the equation, the discretized system is still affected by the sampling time.

To process discretized systems, we often employ the famous z-transform which is defined for a sequence of sampled data $x[k]$ as below:

$$\mathscr{Z}\{x[k]\} = X(z) = \sum_{k=0}^{\infty} x[k] z^{-k} \tag{10}$$

which is a series and converges outside the circle $|z| > R$ with radius R. Compared to the following sampled data

$$x_s(t) = x(t)\delta_T(t) \tag{11}$$

where $x(t) = 0$ for t<0 and

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{12}$$

we obtain

$$\begin{aligned} x_s(t) &= \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) \\ &= \sum_{k=0}^{\infty} x[k]\delta(t-kT) \end{aligned} \tag{13}$$

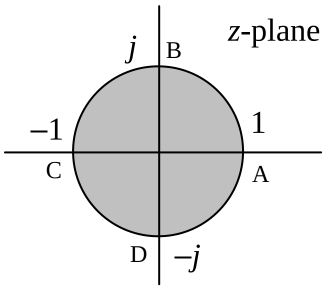
Taking Laplace transform yields

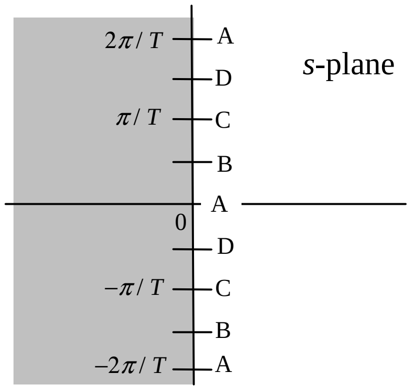
$$X_s(s) = \sum_{k=0}^{\infty} x[k] e^{-ksT} = \sum_{k=0}^{\infty} x[k] \left(e^{sT}\right)^{-k} \tag{14}$$

It is obvious that

$$X_s(s) = X(z)\big|_{z=e^{sT}} \tag{15}$$

The relationship of $z = e^{sT}$ leads to the fact that the left-hand half of the s-plane maps to the interior of the unit circle in the z-plane and the $j\omega$ axis in the s-plane maps to the unit circle in the z-plane, as shown below:





From the figure, we also know that a complete rotation around the unit circle in z-plane is equivalent the frequency ω varies from $-\pi/T$ to π/T. For a continuous system to be stable, it is required that all the poles are located in the left-hand half s-plane. From the mapping, we can conclude that for a discrete system to be stable, all the ploes must be within the unit circle in the z-plane.

Next, let's derive the z-transforms of some commonly used discrete signals and show the related properties of z-transform.

Commonly used discrete signals:

(1) Delta function

$$x[k] = \delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{16}$$

The z-transform is

$$X(z) = \mathcal{Z}\{x[k]\} = \mathcal{Z}\{\delta[k]\} = \sum_{k=0}^{\infty} \delta[k] z^{-k} = \delta[0] z^{-0} = 1 \tag{17}$$

(2) Exponential signal

$$x[k] = a^k, \qquad a>0 \tag{18}$$

The z-transform is

$$X(z) = \mathcal{Z}\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(az^{-1}\right)^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \tag{19}$$

where the condition $|z| > a$ must be satisfied to guarantee the convergence of the transform.

(3) Unit step signal

$$s[k]=1, \qquad \text{for } k=0,1,2,\ldots \tag{20}$$

Since $s[k]=1^k$, form (19) that a=1, we have

$$\mathcal{Z}\{s[k]\}=\sum_{k=0}^{\infty}1^k z^{-k}=\frac{z}{z-1} \tag{21}$$

where $|z|>1$.

(4) Natural exponential signal

$$x[k]=e^{\alpha k}=\left(e^{\alpha}\right)^k, \qquad \alpha\in\Re \tag{22}$$

Since $x[k]=\left(e^{\alpha}\right)^k$, from (19) that $a=e^{\alpha}$, we have

$$\mathcal{Z}\{e^{\alpha k}\}=\frac{z}{z-e^{\alpha}} \tag{23}$$

where $|z|>e^{\alpha}$.

(5) Complex exponential signal

$$e^{j\beta k}=\left(e^{j\beta}\right)^k, \qquad \beta\in\Re \tag{24}$$

The z-transform is

$$\mathcal{Z}\{e^{j\beta k}\}=\sum_{k=0}^{\infty}e^{j\beta k}z^{-k}=\sum_{k=0}^{\infty}\left(e^{j\beta}z^{-1}\right)^k=\frac{z}{z-e^{j\beta}} \tag{25}$$

where $|z|>\left|e^{j\beta}\right|=1$.

(6) Trigonometric signals *cosβk* and *sinβk*

From (25), we have

$$\mathcal{Z}\{e^{j\beta k}\}=\mathcal{Z}\{\cos\beta k\}+j\mathcal{Z}\{\sin\beta k\} \tag{26}$$

and

$$\frac{z}{z-e^{j\beta}}=\frac{z\left(z-e^{-j\beta}\right)}{\left(z-e^{j\beta}\right)\left(z-e^{-j\beta}\right)}=\frac{z^2-z\cos\beta+jz\sin\beta}{z^2-2z\cos\beta+1} \tag{27}$$

Therefore, for the real part we obtain

$$\mathcal{Z}\{\cos\beta k\}=\frac{z^2-z\cos\beta}{z^2-2z\cos\beta+1} \tag{28}$$

and for the imaginary part we obtain

$$\mathcal{Z}\{\sin\beta k\}=\frac{z\sin\beta}{z^2-2z\cos\beta+1} \tag{29}$$

(7) Ramp signal

$$r[k]=k \tag{30}$$

The z-transform is expressed as

$$R(z)=\mathcal{Z}\{r\{k\}\}=\mathcal{Z}\{k\}=\sum_{k=0}^{\infty}kz^{-k} \tag{31}$$

and then

$$\begin{aligned} zR(z)-R(z) &= \sum_{k=0}^{\infty}kz^{-k+1}-\sum_{k=0}^{\infty}kz^{-k} \\ &= \left(1+2z^{-1}+3z^{-2}+4z^{-3}+\cdots\right)-\left(z^{-1}+2z^{-2}+3z^{-3}+\cdots\right) \\ &= 1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+\cdots=\frac{z}{z-1} \end{aligned} \tag{32}$$

After rearranging we have

$$R(z)=\frac{z}{(z-1)^2} \tag{33}$$

A table for z-transform pairs will be shown below:

Pairs of z-transform:

$\delta[k]$	1	(34)
a^k	$\frac{z}{z-a}$	(35)
1	$\frac{z}{z-1}$	(36)
k	$\frac{z}{(z-1)^2}$	(37)
$e^{\alpha k}$	$\frac{z}{z-e^{\alpha}}$	(38)
$e^{j\beta k}$	$\frac{z}{z-e^{j\beta}}$	(39)
$\cos\beta t$	$\frac{z^2-z\cos\beta}{z^2-2z\cos\beta+1}$	(40)
$\sin\beta t$	$\frac{z\sin\beta}{z^2-2z\cos\beta+1}$	(41)

Some important properties related to the z-transform have been widely applied to discrete signal processing and will be further discussed below:

(1) Linearity

$$\mathcal{Z}\{ax[k]+by[k]\}=a\mathcal{Z}\{x[k]\}+b\mathcal{Z}\{y[k]\}=aX(z)+bY(z) \tag{42}$$

(2) Time-shifting

Consider an m-delayed signal $x[k–m]$, whose z-transform is

$$\mathcal{Z}\{x[k-m]\}=\sum_{k=0}^{\infty}x[k-m]z^{-k}=\sum_{p=-m}^{\infty}x[p]z^{-(p+m)} \tag{43}$$

$$=z^{-m}\left(\sum_{p=-m}^{-1}x[p]z^{-p}+\sum_{p=0}^{\infty}x[p]z^{-p}\right)$$

$$=z^{-m}\left(\sum_{q=1}^{m}x[-q]z^{q}+X(z)\right)$$

If $x[k]=0$ for $k<0$, then

$$\mathcal{Z}\{x[k-m]\}=z^{-m}X(z) \tag{44}$$

Consider an m-advanced signal $x[k+m]$, whose z-transform is

$$\mathcal{Z}\{x[k+m]\}=\sum_{k=0}^{\infty}x[k+m]z^{-k}=\sum_{p=m}^{\infty}x[p]z^{-(p-m)} \tag{45}$$

$$=z^{m}\left(\sum_{p=0}^{\infty}x[p]z^{-p}-\sum_{p=0}^{m-1}x[p]z^{-p}\right)$$

$$=z^{m}\left(X(z)-\sum_{p=0}^{m-1}x[p]z^{-p}\right)$$

(3) Premultiplying a^{-k} to $x[k]$

$$\mathcal{Z}\{a^{-k}x[k]\}=\sum_{k=0}^{\infty}a^{-k}x[k]z^{-k}=\sum_{k=0}^{\infty}x[k](az)^{-k}=X(az) \tag{46}$$

From (40) and (46), we have

$$\mathcal{Z}\{b^{k}\cos\beta k\}=\mathcal{Z}\left\{\left(\frac{1}{b}\right)^{-k}\cos\beta k\right\}=\frac{z^{2}-bz\cos\beta}{z^{2}-2bz\cos\beta+b^{2}} \tag{47}$$

From (41) and (46), we have

$$\mathcal{Z}\{b^{k}\sin\beta k\}=\mathcal{Z}\left\{\left(\frac{1}{b}\right)^{-k}\sin\beta k\right\}=\frac{bz\sin\beta}{z^{2}-2bz\cos\beta+b^{2}} \tag{48}$$

(4) Premultiplying k^{n} to $x[k]$

If we differentiate $X(z)$ with respect to z, then we have

$$\frac{d}{dz}X(z)=\frac{d}{dz}\sum_{k=0}^{\infty}x[k]z^{-k}=\sum_{k=0}^{\infty}-kx[k]z^{-k-1} \tag{49}$$

which results in

$$\mathcal{Z}\{kx[k]\}=\sum_{k=0}^{\infty}kx[k]z^{-k}=-z\frac{d}{dz}X(z) \tag{50}$$

Consequently, we obtain

$$\mathcal{Z}\{k^n x[k]\}=\left(-z\frac{d}{dz}\right)^n X(z) \tag{51}$$

(5) Initial value theorem

From the definition of z-transform in (10), i.e.,

$$X(z)=\sum_{k=0}^{\infty}x[k]z^{-k}=x[0]+x[1]z^{-1}+x[2]z^{-2}+\cdots \tag{52}$$

the initial $x[0]$ can be achieved as below:

$$x[0]=\lim_{z\to\infty}X(z) \tag{53}$$

(6) Final value theorem

Assume $x[k]$ is stable and converges to a constant or zero, then

$$\mathcal{Z}\{x[k+1]-x[k]\}=\sum_{\substack{k=0\\N\to\infty}}^{N}x[k+1]z^{-k}-\sum_{\substack{k=0\\N\to\infty}}^{N}x[k]z^{-k} \tag{54}$$

If $z\to 1$, then

$$zX(z)-x[0]-X(z)=\sum_{\substack{k=0\\N\to\infty}}^{N-1}x[k+1]-\sum_{\substack{k=0\\N\to\infty}}^{N-1}x[k]=x[N]_{N\to\infty}-x[0] \tag{55}$$

i.e.,

$$x[\infty]=\lim_{z\to 1}(z-1)X(z) \tag{56}$$

(7) Convolution

Define the convolution of $h[k]$ and $u[k]$ as below:

$$h[k]*u[k]=\sum_{m=0}^{\infty}h[k-m]u[m] \tag{57}$$

where $h[k]=u[k]=0$ for $k<0$. The z-transform of $h[k]*u[k]$ is then obtained as

$$\mathcal{Z}\{h[k]*u[k]\}=\sum_{k=0}^{\infty}(h[k]*u[k])z^{-k}=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}h[k-m]u[m]z^{-k} \tag{58}$$

It is further arranged as

$$\mathcal{Z}\{h[k]*u[k]\}=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}h[k-m]z^{-(k-m)}\right)u[m]z^{-m} \tag{59}$$

where

$$\sum_{k=0}^{\infty}h[k-m]z^{-(k-m)}=\sum_{p=-m}^{\infty}h[p]z^{-p}=\sum_{p=0}^{\infty}h[p]z^{-p}=H(z) \tag{60}$$

Therefore, from (58) we have

$$\mathcal{Z}\{h[k]*u[k]\}=H(z)\sum_{m=0}^{\infty}u[m]z^{-m}=H(z)U(z) \tag{61}$$

Next, we will solve the discrete system by the use of z-transform.