13. Sampling and Modulation

The concept of signal sampling enables us to extend the field of continuous signals to the field of discrete signals. First, let's check the ideal impulse sampling by an impulse train shown as below and expressed as

$$x_s(t) = x(t)\delta_T(t) \tag{1}$$

where

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{2}$$

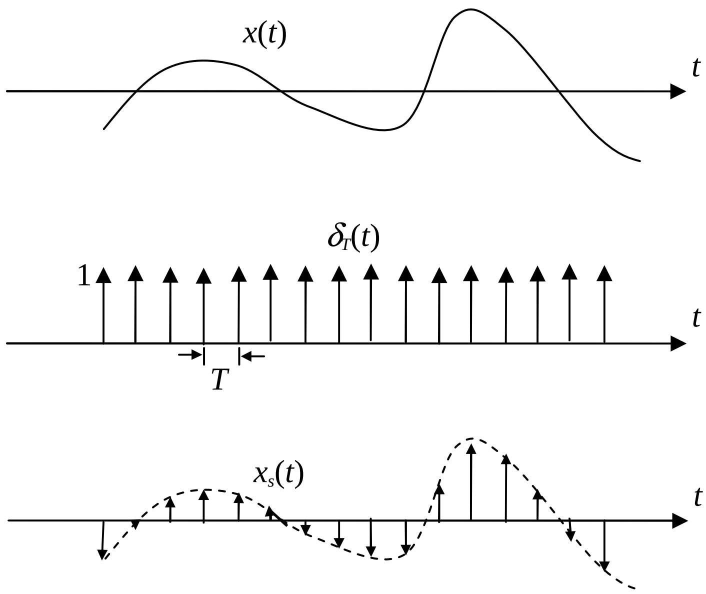
where $c_k = \dfrac{1}{T}$ and $\omega_0 = \dfrac{2\pi}{T}$. Hence, we have

$$x_s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(t) e^{jk\omega_0 t} \tag{3}$$

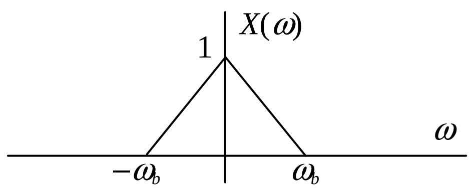
whose. Fourier transform is

$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0) \tag{4}$$

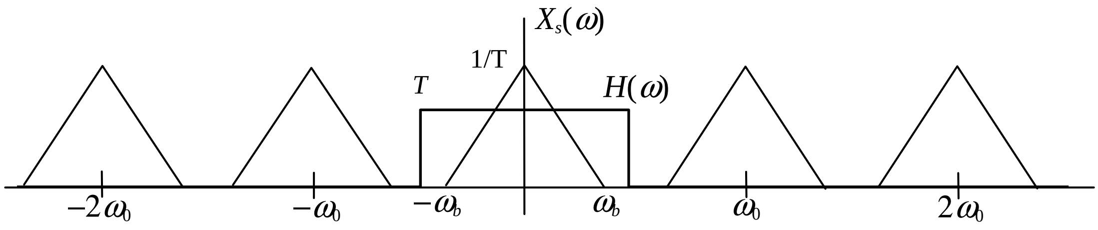
where $X_s(\omega)$ and $X(\omega)$ are the Fourier transforms of $x_s(t)$ and $x(t)$, respectively.



Assume $X(\omega)$ is band limited to $0 \le \omega \le \omega_b$ as shown in the figure, where ω_b is the highest frequency of the signal. From (4) we know that $X_s(\omega)$ consists of



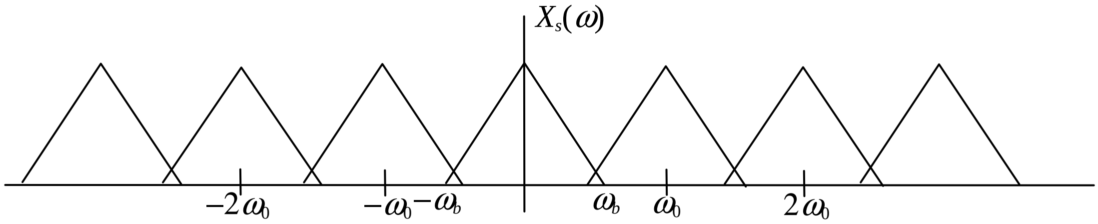
the original spectrum $X(\omega)$ and an infinite number of scaled and shifted versions of the original spectrum, which are $\frac{1}{T}X(\omega-k\omega_0)$ for $k\in Z$. If we want to recover the original signal $x(t)$ from the modulated signal $x_s(t)$, it is required that none of the shifted spectral components overlap each other, depicted as below:



which leads to the important condition:

$$\omega_0 \geq 2\omega_b \tag{5}$$

called the Sannon's Sampling theorem. Then, the original signal $x(t)$ can be recovered from $x_p(t)$ by a suitable lowpass filter like $H(\omega)$ in the figure. Note that the minimum sampling frequency of ω_0 to recover the original signal is $2\omega_b$, called the Nyquist rate. On the other hand, if $0<\omega_0<2\omega_b$ then overlap happens as shown below:



The phenomenon of overlapping is called aliasing and makes the recovery of $x(t)$ impossible.

In practice, we use the signals obtained by modulating the continuous signals by a pulse train shown as below and expessed as

$$x_p(t) = x(t)p(t) \tag{6}$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{7}$$

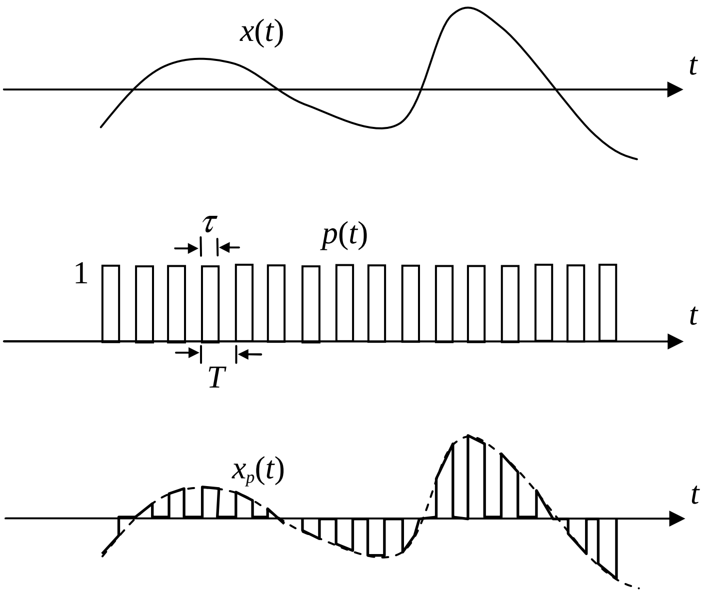
and $\omega_0 = \frac{2\pi}{T}$. Hence, we have

$$x_p(t) = x(t)\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k x(t) e^{jk\omega_0 t} \qquad (8)$$

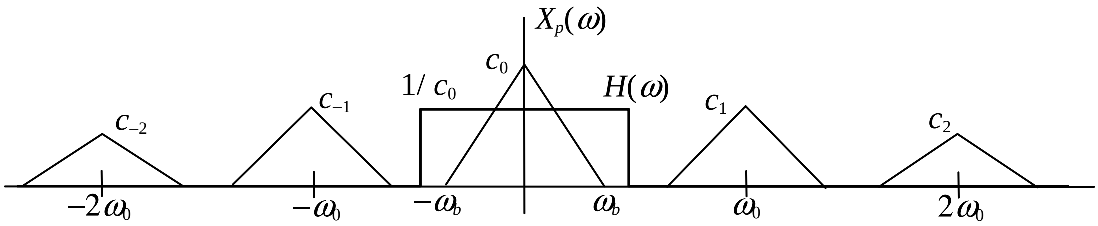
whose Fourier transform is

$$X_p(\omega) = \sum_{k=-\infty}^{\infty} c_k X(\omega - k\omega_0) \qquad (9)$$

where $X_p(\omega)$ and $X(\omega)$ are the Fourier transforms of $x_p(t)$ and $x(t)$, respectively.



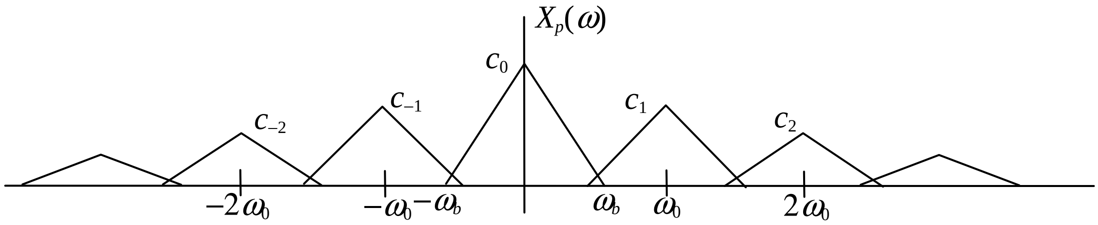
Assume $X(\omega)$ is band limited to $0\le\omega\le\omega_b$ as shown before, whose highest frequency is ω_b. From (9) we know that $X_p(\omega)$ consists of the original spectrum $X(\omega)$ and an infinite number of scaled and shifted versions of the original spectrum, which are $c_m X(\omega - k\omega_0)$ for $k\in Z$. If we want to recover the original signal $x(t)$ from the modulated signal $x_p(t)$, it is required that none of the shifted spectral components overlap each other, depicted as below:



which leads to the Sannon's Sampling theorem:

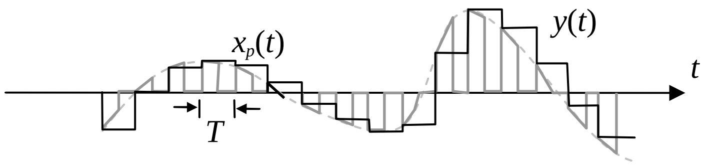
$$\omega_0 \ge 2\omega_b \qquad (10)$$

Then, the original signal $x(t)$ can be recovered from $x_p(t)$ by a suitable lowpass filter like $H(\omega)$ in the figure. Similarly, the minimum sampling frequency of ω_0 to recover the original signal is the Nyquist rate $2\omega_b$. On the other hand, if $0<\omega_0<2\omega_b$ then overlap, or aliasing, happens and makes the recovery of $x(t)$ impossible.

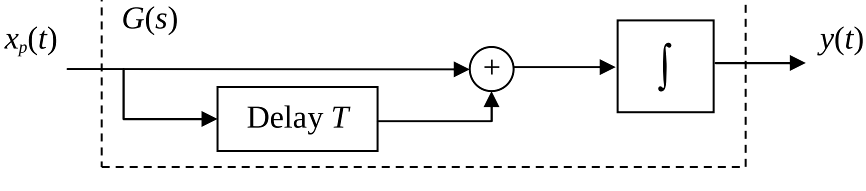


The process of signal recovery can be implemented by the ideal lowpass filter $H(\omega)$. However, it is impractical. Instead, a lot of approximate lowpass filters are used to recover the signal, such as the holding circuit. Here, we will introduce the simplest one, called the zero-order holding circuit.

The function of the zero-order holding circuit is to hold the value of sampled signal $x_p(t)$ for T seconds, as shown in the following figure:



The holding circuit maintains the value of the pulse at the beginning to an interval T, which makes the sampled signal become a continuous signal approximated to the original signal $x(t)$. The holding zero-order holding circuit is often implemented as below:



whose impulse response is

$$G(s)=\frac{1-e^{-sT}}{s} \tag{10}$$

and the Fourier transform is

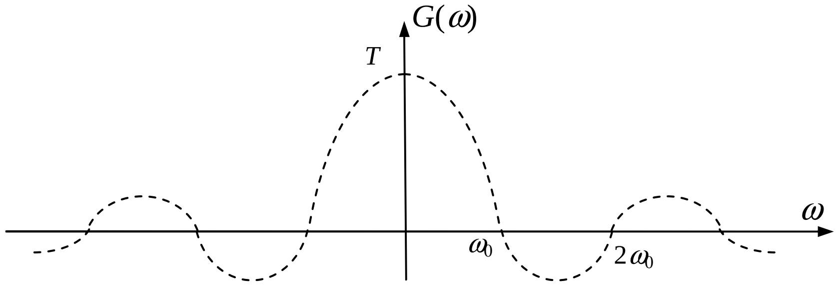
$$G(j\omega)=\frac{1-e^{-j\omega T}}{j\omega} \tag{11}$$

i.e.,

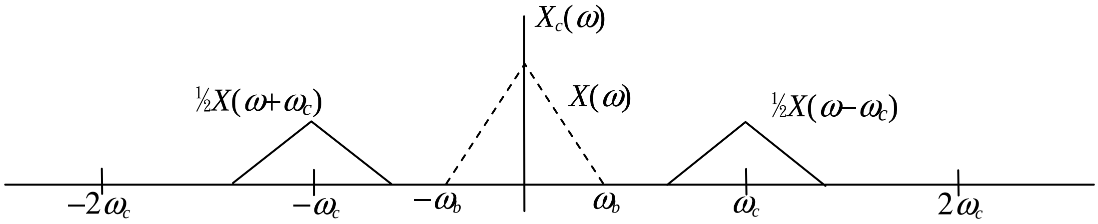
$$|G(j\omega)|=\left|\frac{\sqrt{2-2\cos\omega T}}{\omega}\right|=T\,\frac{\sin(\omega T/2)}{\omega T/2}=T\,sinc\left(\frac{\omega T}{2}\right) \tag{12}$$

$$\angle G(j\omega)=-\frac{\omega T}{2} \tag{13}$$

The magnitude frequency spectrum is given in the following.



In communication systems, modulation is one of the important techniques. Here, we will introduce the basic concept of modulation and demodulation by the use of a sinusoidal carrier.



Assume $x(t)$ is an information-bearing signal whose frequency is band-limited by $0\leq\omega\leq\omega_b$. Let $x(t)$ be sent from a transmiter modulated by a sinusoidal carrier $\cos\omega_c t$ and the modulated signal is expressed as

$$x_c(t)=x(t)\cos\omega_c t \tag{14}$$

where the carrier frequency ω_c is chosen greater than ω_b. The modulated signal can be further written as

$$x_c(t)=\frac{1}{2}x(t)\left(e^{j\omega_c t}+e^{-j\omega_c t}\right)=\frac{1}{2}x(t)e^{j\omega_c t}+\frac{1}{2}x(t)e^{-j\omega_c t} \tag{15}$$

whose Fourier transform is

$$X_c(\omega) = \frac{1}{2}X(\omega - \omega_c) + \frac{1}{2}X(\omega + \omega_c) \tag{16}$$

as depicted in the Figure.

At the receiver, the information contained in $x(t)$ can be recovered through demodulation by multiplying the same sinusoidal carrier as below:

$$y(t) = x_c(t)\cos\omega_c t = x(t)\cos^2\omega_c t = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t \tag{17}$$

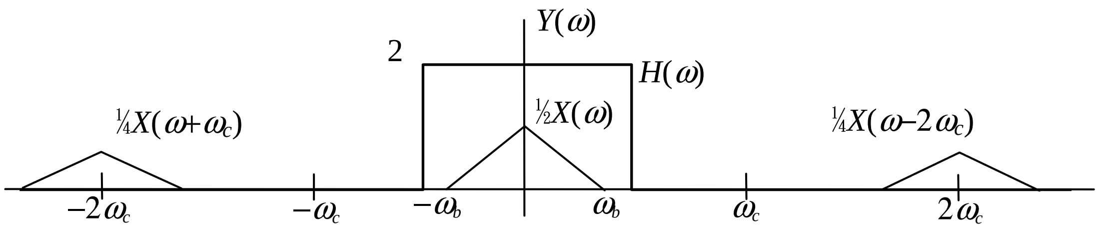
i.e.,

$$y(t) = \frac{1}{2}x(t) + \frac{1}{4}x(t)e^{j2\omega_c t} + \frac{1}{4}x(t)e^{-j2\omega_c t} \tag{18}$$

whose Fourier transform is

$$Y(\omega) = \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 2\omega_c) + \frac{1}{4}X(\omega + 2\omega_c) \tag{19}$$

as depicted in the Figure.



It is clear that the information-bearing signal $x(t)$ can be recovered from $y(t)$ at the receiver by the use of an ideal lowpass filter $H(\omega)$ as shown in the figure.