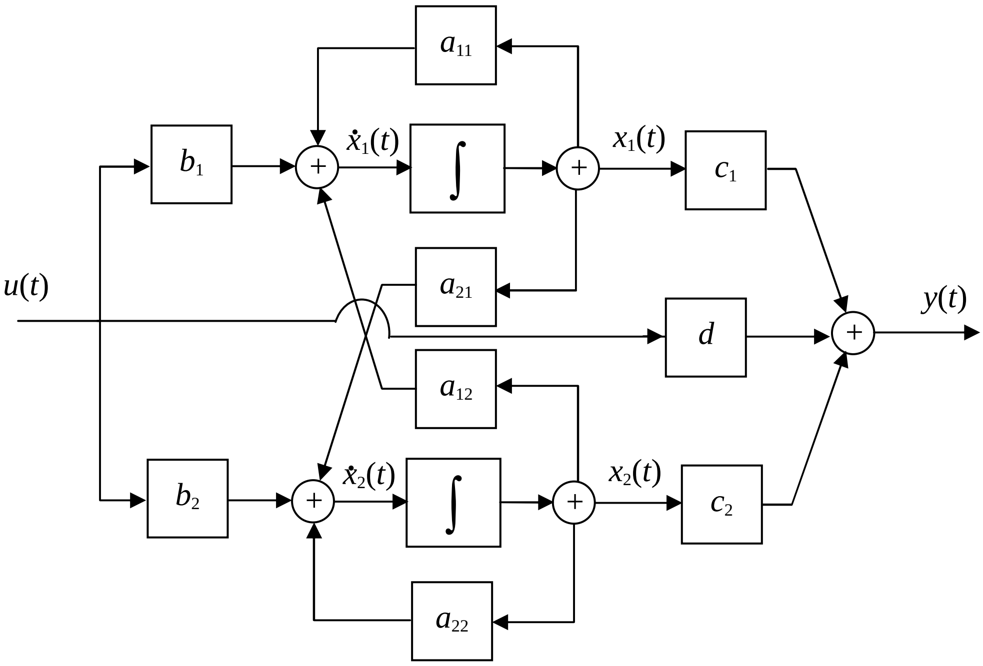
12. System in State-Space Description

It has been introduced that an LTI system can be expressed by an ordinary differential equation to describe the relation between input and output. Usually, we say that the system is in input-output description. However, the input-output description is unavailable to express the whole features of a system. If a system is known all in detail, not just the relation between input and output, we often write it in state-space description.

For simplicity, we only present a 2nd order system in state-space description here and all the properties discussed below are also available for higher order systems.



In general, an LTI system in state space description is expressed by two 1st order differential equations as below:

$$\dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t), \qquad x_1(0)=x_{10} \tag{1}$$

$$\dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t), \qquad x_2(0)=x_{20} \tag{2}$$

and its output eqation is

$$y(t) = c_1x_1(t) + c_2x_2(t) + du(t) \tag{3}$$

The block diagram is shown in the above figure.

For convenience, the above 2nd LTI system is often rearranged as a vector form, where the state equation is

$$\underbrace{\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}}_{\dot{\boldsymbol{x}}(t)}=\underbrace{\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}_{\boldsymbol{A}}\cdot\underbrace{\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}}_{\boldsymbol{x}(t)}+\underbrace{\begin{bmatrix}b_1\\ b_2\end{bmatrix}}_{\boldsymbol{b}}u(t),\qquad \underbrace{\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}}_{\boldsymbol{x}(0)}=\underbrace{\begin{bmatrix}x_{10}\\ x_{20}\end{bmatrix}}_{\boldsymbol{x}_0} \tag{4}$$

and the output equation is

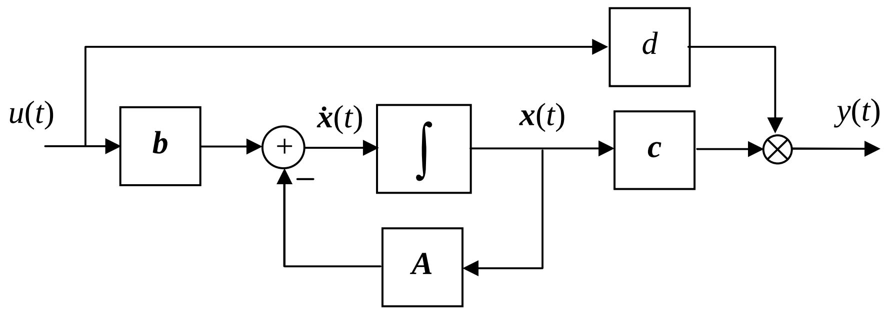
$$y(t)=\underbrace{\begin{bmatrix}c_1 & c_2\end{bmatrix}}_{\boldsymbol{c}}\cdot\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+du(t) \tag{5}$$

Both are rewritten as below:

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\cdot\boldsymbol{x}(t)+\boldsymbol{b}u(t),\qquad \boldsymbol{x}(0)=\boldsymbol{x}_0 \tag{6}$$

$$y(t)=\boldsymbol{c}\cdot\boldsymbol{x}(t)+du(t) \tag{7}$$

The followign figure shows the block diagram correspondingly.



It is known that the system state $\boldsymbol{x}(t)$ can be solved from the state equation (6) and expressed as

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}t}\,\boldsymbol{x}_0+\int_0^t e^{\boldsymbol{A}(t-\tau)}\,\boldsymbol{b}\,u(\tau)d\tau \tag{8}$$

and then the output is

$$y(t)=\boldsymbol{c}\boldsymbol{x}(t)+du(t)=\boldsymbol{c}e^{\boldsymbol{A}t}\boldsymbol{x}_0+\int_0^t \boldsymbol{c}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)d\tau+du(t) \tag{9}$$

Although the state vector $\boldsymbol{x}(t)$ can be determined by directly calculating (8), it is often not so easy to get the solution numerically. Fortunately, we can apply the software MATLAB to solve differential equations. Let's consider the following equation as an example:

$$\dot{x}_1(t)=x_2(t),\qquad x_1(0)=1 \tag{10}$$

$$\dot{x}_2(t)=-3x_1(t)-4x_2(t)+u(t),\qquad x_2(0)=0 \tag{11}$$

$$y(t)=2x_1(t)-x_2(t) \tag{12}$$

If the input is $u(t)=sin(cos(5t))$, then the system response $y(t)$ can be obained by the instruction ode45 in MATLAB.

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Create m-file: second.m
  function dx=second(t,x)
  dx=zeros(2,1);
  dx(1)=x(2);
  dx(2)=-3*x(1)-4*x(2)+sin(cos(5*t));

  >> % key in the following instructions
  >> [t,x]=ode45(@second,[0:0.01:6],[1 0])
  >> y=2*x(:,1)-x(:,2);
  >> plot(t,y); xlabel('t'); ylabel('y(t)')
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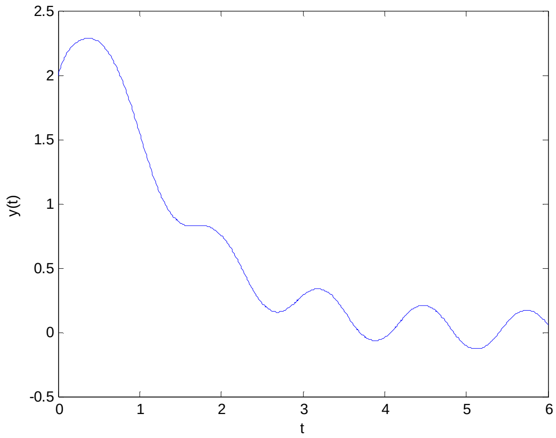

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In fact, based on the state space description, an LTI system can be also represented by the input-output description, not related to the state vector $\boldsymbol{x}(t)$. Now, let's show the procedure to derive the input-output description from the state space description. First, find the characteristic equation of the system matrix $\boldsymbol{A}$, which is defined as

$$p(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = 0 \tag{13}$$

i.e.,

$$p(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0 \tag{14}$$

where $p(\lambda)$ is called the characteristic polynomial. Let $a_1 = -(a_{11} + a_{22})$ and $a_0 = a_{11}a_{22} - a_{12}a_{21}$, then the characteristic polynomial becomes

$$p(\lambda) = \lambda^2 + a_1\lambda + a_0 \tag{15}$$

According to the Cayley-Hamilton theoreom, we have

$$p(\boldsymbol{A}) = \boldsymbol{A}^2 + a_1\,\boldsymbol{A} + a_0\,\boldsymbol{I} = \boldsymbol{0} \tag{16}$$

Then, calculate the following equation:

$$\begin{aligned}
&\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) \\
&= \boldsymbol{c}(\ddot{\boldsymbol{x}}(t) + a_1\dot{\boldsymbol{x}}(t) + a_0\boldsymbol{x}(t)) + d\ddot{u}(t) + a_1 d\dot{u}(t) + a_0 du(t) \\
&= \boldsymbol{c}\left(\boldsymbol{A}^2\boldsymbol{x}(t) + \boldsymbol{Ab}u(t) + \boldsymbol{b}\dot{u}(t) + a_1(\boldsymbol{Ax}(t) + \boldsymbol{b}u(t)) + a_0\boldsymbol{x}(t)\right) \\
&\qquad\qquad + d\ddot{u}(t) + a_1 d\dot{u}(t) + a_0 du(t) \\
&= \boldsymbol{c}\left(\boldsymbol{A}^2 + a_1\boldsymbol{A} + a_0\boldsymbol{I}\right)\boldsymbol{x}(t) + d\ddot{u}(t) + (\boldsymbol{cb} + a_1 d)\dot{u}(t) \\
&\qquad\qquad + (\boldsymbol{cAb} + a_1\boldsymbol{cb} + a_0 d)u(t)
\end{aligned} \tag{17}$$

From (16) and (17), we have

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0 u(t) \tag{18}$$

where $b_2 = d$, $b_1 = \boldsymbol{cb}$ and $b_0 = \boldsymbol{cAb} + a_1\boldsymbol{cb} + a_0 d$. This leads to the input-output description.

P.1 Consider the following system:

$$\dot{x}_1(t) = x_2(t), \qquad x_1(0)=1$$

$$\dot{x}_2(t) = -5x_1(t) - 2x_2(t), \qquad x_2(0)=1$$

$$y(t) = x_1(t) + 3x_2(t)$$

where the input is $u(t)=sin(cos(3t))$. Solve the output $y(t)$ by MATLAB.