11. System Realization

After a filter is designed, one question is raised: How to implement the filter? In system engineering, we often use the so-called system realiztion to implement it. Consider a filter is designed as an LTI system described by the following equation:

$$\begin{aligned} &y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t) \\ &=b_mu^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+b_1\dot{u}(t)+b_0u(t) \end{aligned} \quad (1)$$

which of course is stable. By neglecting the initial conditions, we obtain the frequency response as

$$Y(s)=H(s)U(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s) \quad (2)$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of input $u(t)$ and output $y(t)$, and the transfer function is

$$H(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{b_m(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_m)} \quad (3)$$

where z_j, j=1,2,...,m, are the zeros and p_k, k=1,2,...,n, are the poles located in the left-half complex plane.

Let's consider $n>m$ first. It is known that there are an infinite number of ways to realize the system (2) and the most fundamental is to choose state variables as below:

$$X_k(s)=\frac{s^{k-1}}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s),\quad k=1,2,\ldots,n \quad (4)$$

or

$$X_1(s)=\frac{1}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}U(s) \quad (5)$$

$$X_k(s)=s^{k-1}X_1(s)=s^{k-2}X_2(s)=\cdots=sX_{k-1}(s),\ k=2,3,\ldots,n \quad (6)$$

From (5), we have

$$s^nX_1(s)+a_{n-1}s^{n-1}X_1(s)+\cdots+a_1sX_1(s)+a_0X_1(s)=U(s) \quad (7)$$

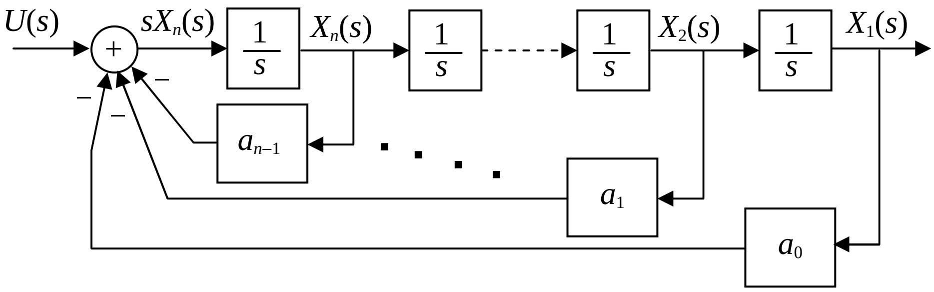
and further from (6) we obtain

$$sX_n(s)+a_{n-1}X_n(s)+\cdots+a_1X_2(s)+a_0X_1(s)=U(s) \quad (8)$$

i.e.,

$$sX_n(s) = -a_0 X_1(s) - a_1 X_2(s) - \cdots - a_{n-1} X_n(s) + U(s) \tag{9}$$

The block diagram is shown as below;



Let $x_k(t) = \mathcal{L}^{-1}\{X_k(s)\}$, then from (6) and (9) we have

$$\dot{x}_{k-1}(t) = x_k(t), \qquad k=2,3,\ldots,n \tag{10}$$

$$\dot{x}_n(t) = -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t) \tag{11}$$

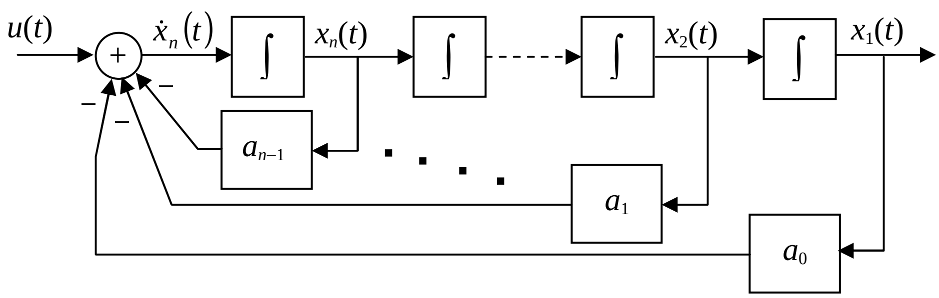
both leading to the following state equation

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \quad\vdots \\ \dot{x}_{n-1}(t) = x_n(t) \\ \dot{x}_n(t) = -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t) \end{cases} \tag{12}$$

or in the matrix form as

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \vdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix}}_{\boldsymbol{x}(t)} \tag{13}$$

where $\boldsymbol{x}(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_{n-1}(t) \quad x_n(t)]^T$ is called the state vector. Similarly, the corresponding block diagram is shown as below:



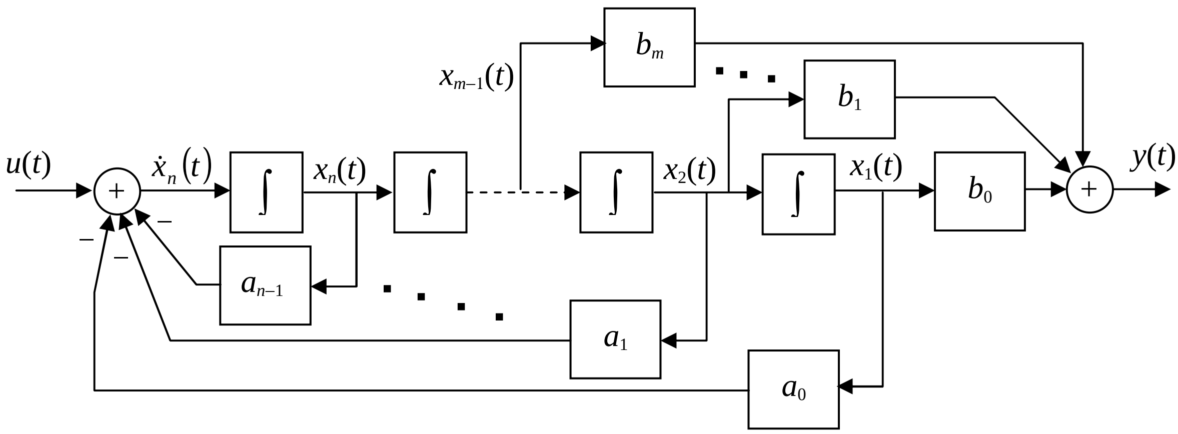
From (2) and (4), the output equation can be derived as

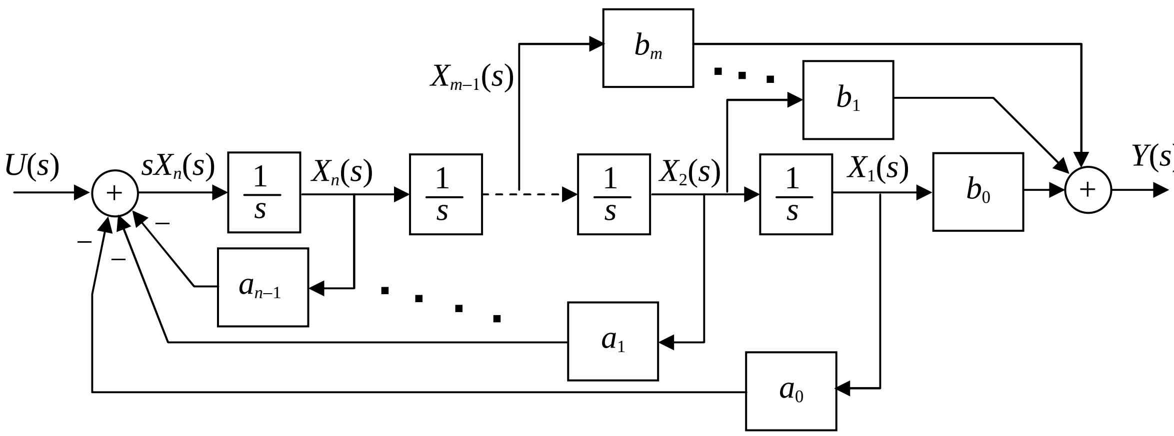
$$Y(s) = b_m X_{m-1}(s) + \cdots + b_2 X_3(s) + b_1 X_2(s) + b_0 X_1(s) \tag{14}$$

i.e.,

$$y(t) = b_m x_{m-1}(t) + \cdots + b_2 x_3(t) + b_1 x_2(t) + b_0 x_1(t) \tag{15}$$

As a result, the total system can be implemented in time domain or in frequency domain as below:





In the case of *n*=*m*, such as highpass filter, the transfer function is given as

$$H(s)=\frac{b_n s^n+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \tag{16}$$

Rearranging it yields

$$H(s)=b_n+\frac{\beta_{n-1}s^{n-1}+\cdots+\beta_1 s+\beta_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \tag{17}$$

where

$$\beta_k=b_k-b_n a_k, \qquad k=0,1,2,\ldots,n-1 \tag{18}$$

Therefore,

$$Y(s)=H(s)U(s)=\left(b_n+\frac{\beta_{n-1}s^{n-1}+\cdots+\beta_1 s+\beta_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}\right)U(s) \tag{19}$$

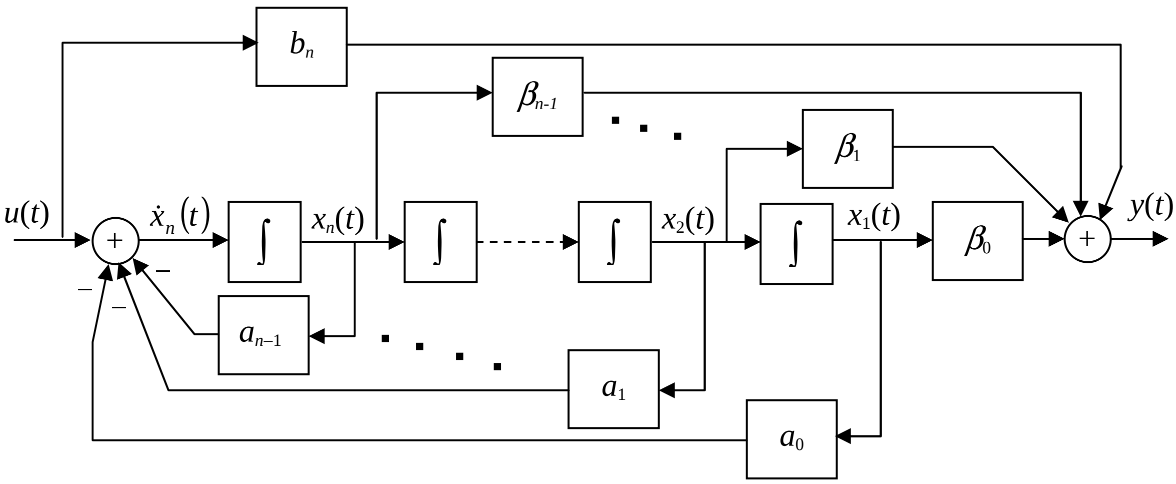
i.e.,

$$Y(s)=b_n U(s)+\beta_{n-1}X_n(s)+\cdots+\beta_2 X_3(s)+\beta_1 X_2(s)+\beta_0 X_1(s) \tag{20}$$

or

$$y(t)=b_n u(t)+\beta_{n-1}x_n(t)+\cdots+\beta_2 x_3(t)+\beta_1 x_2(t)+\beta_0 x_1(t) \tag{21}$$

The block diagram is



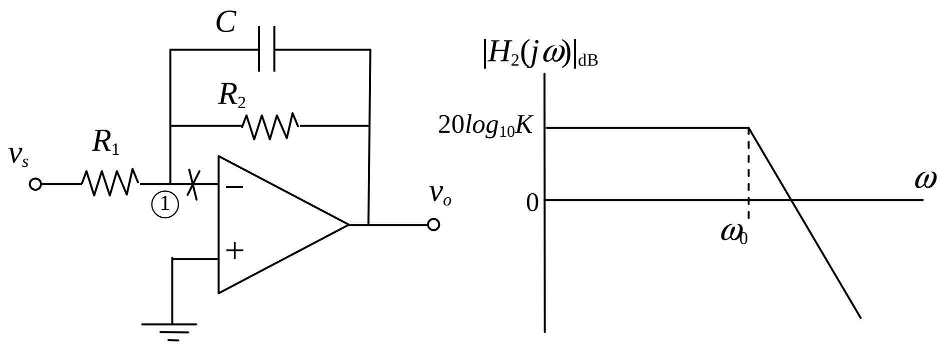
In addition to the above realization, the system can be changed into series, parallel, or a lot of different forms. Most importantly, all of them can be realized by the use of OpAmps.

[補充教材] 主動式濾波器

一般由 *RLC* 等被動元件所組合而成的濾波器，稱為被動式濾波器，其特性會隨著外接的負載改變，常造成使用上的困擾，為了改進此缺點，具有高輸入電阻與低輸出電阻特性的 OP-amp 便成了最佳的選擇，因為它能夠不受負載變化的影響，此外 OP-amp 為主動式元件，所以將 OP-amp 所設計的濾波器，稱為主動式濾波器。

■ **一階低通濾波器**

最簡單的主動式一階低通濾波器如圖所示，若單純從電容的阻抗來看，可知在高頻時電容 C 可視為短路，使得輸出端的電壓 $v_o(t)$與節點①的電壓相同，由於節點①為虛擬接地，所以 $v_o(t)=0$，即高頻的輸入訊號會被濾除；而在低頻時電容 C 可視為開路，此時整個電路形同反相放大器，使得輸出訊號與輸入訊號反相，但大小成正比，即輸入訊號可以順利通過且放大，根據以上的觀察與分析，可以知道它確實是一個低通濾波器。



推導此低通濾波器之轉移函數如下：

$$\frac{V_s(s)}{R_1}=\frac{-V_o(s)}{R_2}+\frac{-V_o(s)}{1/sC} \tag{1}$$

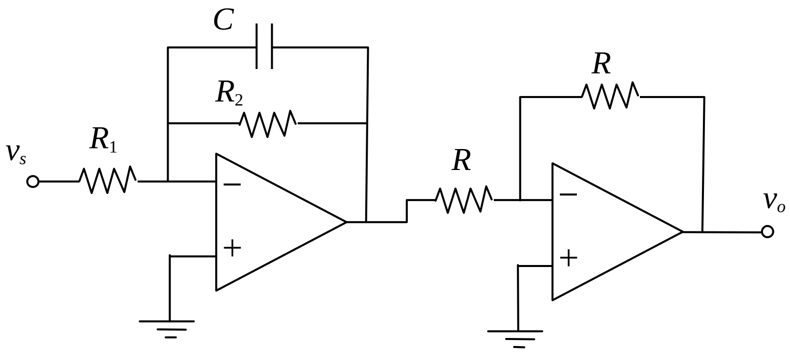
進一步整理後可得

$$H(s)=\frac{V_o(s)}{V_s(s)}=-\frac{R_2}{R_1}\left(\frac{1}{1+sR_2C}\right)=-\frac{K}{1+\frac{s}{\omega_0}} \tag{2}$$

其中增益常數 $K=\frac{R_2}{R_1}$，截止頻率 $\omega_0=\frac{1}{R_2C_2}$ ，其頻率響應$|H(j\omega)|$如上圖所示，屬

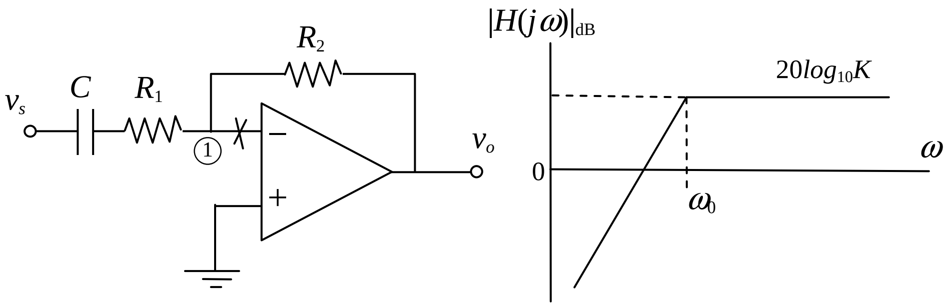
於低通濾波器，低頻的放大倍率為 $K=\frac{R_2}{R_1}$，通常取 $K>1$，即 $R_2>R_1$，故 $|H(j0)|_{dB}=20log_{10}K>0$。

若是想要設計輸出訊號與輸入訊號同相的低通濾波器時，只需要再接上一個單位增益的反相放大器即可，如下圖所示。



■ **一階高通濾波器**

最簡單的主動式一階高通濾波器如下圖所示：

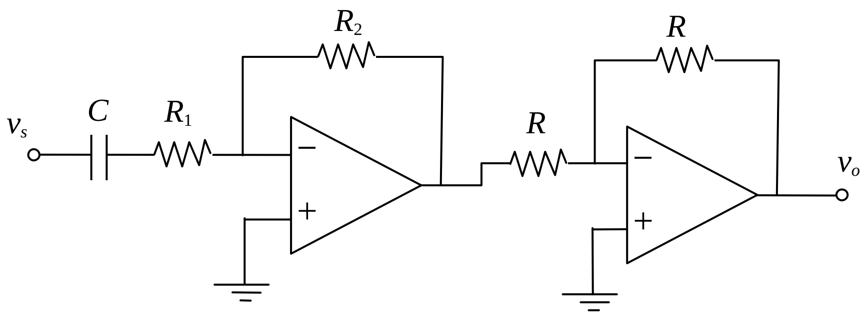


它是在電阻 R_1 上串聯一個電容 C，同樣地可得此電路之轉移函數如下；

$$H(s)=\frac{V_o(s)}{V_s(s)}=-\frac{R_2}{R_1+\frac{1}{sC}}=-\frac{K(s/\omega_0)}{1+\frac{s}{\omega_0}} \tag{3}$$

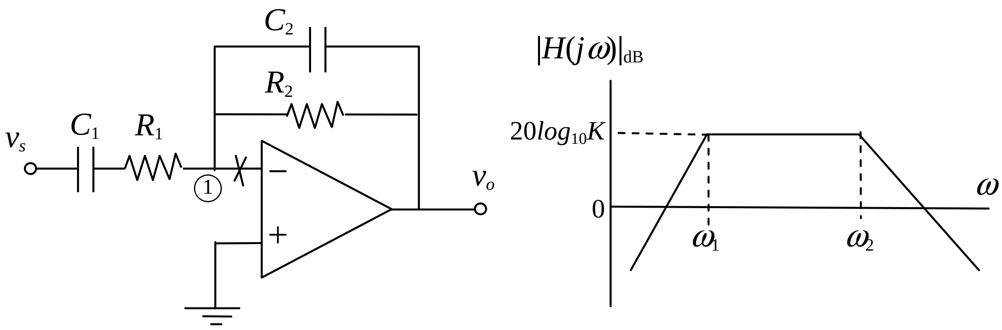
其中增益常數 $K=\frac{R_2}{R_1}$，截止頻率 $\omega_0=\frac{1}{R_1C}$，其頻率響應 $|H(j\omega)|$ 如上圖所示，屬於高通濾波器，放大倍率為 K，通常取 $K>1$，即 $R_2>R_1$，故 $|H(j\infty)|_{dB}=20log_{10}K>0$。

若是想要設計輸出訊號與輸入訊號同相的高通濾波器時，仍然只需要再接上一個單位增益的反相放大器即可，如下圖所示：



■ 二階帶通濾波器

主動式二階帶通濾波器如下圖所示：



它在反相放大器的電阻 R_1 上串聯一個電容 C_1，同時在電阻 R_2 上並聯一個電容 C_2。其轉移函數為

$$H(s)=\frac{V_o(s)}{V_s(s)}=-\frac{1}{\left(R_1+\frac{1}{sC_1}\right)\left(\frac{1}{R_2}+sC_2\right)}$$
$$=-\frac{K(s/\omega_1)}{(1+s/\omega_1)(1+s/\omega_2)} \qquad (4)$$

其中增益常數 $K=\frac{R_2}{R_1}$，截止頻率 $\omega_1=\frac{1}{R_1C_1}$ 與 $\omega_2=\frac{1}{R_2C_2}$，取 $\omega_2>\omega_1$，則(4)可改寫為

$$H(s)=-\left(\frac{K(s/\omega_1)}{1+s/\omega_1}\right)\left(\frac{1}{1+s/\omega_2}\right) \qquad (5)$$

此式表示 $H(s)$是由兩個一階濾波器串接而成，其中 $\frac{K(s/\omega_1)}{1+s/\omega_1}$ 為高通濾波器，$\frac{1}{1+s/\omega_2}$ 為低通濾波器，其頻率響應如上圖所示。