10. Signal Filtering

Signal filtering is an important function for a practical system, especially when the system is operated around a noisy environment. In system engineering to eliminate external noises, the filter is often designed as an LTI system described by the following equation:

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1\dot{y}(t)+a_0y(t) \tag{1}$$
$$=b_mu^{(m)}(t)+b_{m-1}u^{(m-1)}(t)+\cdots+b_1\dot{u}(t)+b_0u(t)$$

which is stable and thus the initial conditions can be neglected. As a result, the frequency response is given as

$$Y(s)=H(s)U(s) \tag{2}$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of input $u(t)$ and output $y(t)$. The transfer function of the system is

$$H(s)=\frac{b_ms^m+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{b_m(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_m)} \tag{3}$$

where p_k, k=1,2,...,n, are the poles and z_j, j=1,2,...,m, are the zeros. To guarantee the system stability, all the poles p_k should be located in the left-half complex plane. Let $s=j\omega$, then (2) becomes

$$Y(j\omega)=H(j\omega)U(j\omega) \tag{4}$$

where $U(j\omega)$ and $Y(j\omega)$ are the Fourier transforms of $u(t)$ and $y(t)$. It is known that

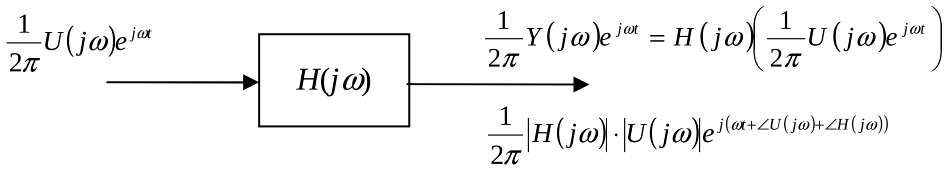
$$u(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}U(j\omega)e^{j\omega t}d\omega=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}U(j\omega)e^{j\omega t}\right)d\omega \tag{5}$$

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j\omega)e^{j\omega t}d\omega=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}Y(j\omega)e^{j\omega t}\right)d\omega \tag{6}$$

which implies the input $u(t)$ and output $y(t)$ are respectively formed by an infinite number of components $\frac{1}{2\pi}U(j\omega)e^{j\omega t}$ and $\frac{1}{2\pi}Y(j\omega)e^{j\omega t}$ at frequency ω. From (4), it is true that

$$\frac{1}{2\pi}Y(j\omega)e^{j\omega t}=H(j\omega)\left(\frac{1}{2\pi}U(j\omega)e^{j\omega t}\right) \tag{7}$$

which is depicted in the following figure. Obviously, if $|H(j\omega)|$ <<1, then the output component at ω is highly decreased. That means the system filter out the input component at ω.



From (4), the frequency response in magnitude can be expressed as the following form

$$|Y(j\omega)|=|H(j\omega)||W(j\omega)| \quad (8)$$

i.e.,

$$|Y(j\omega)|_{dB}=|H(j\omega)|_{dB}+|W(j\omega)|_{dB} \quad (9)$$

where $|\bullet|_{dB}=20log_{10}|\bullet|$, and the frequency response in phase as

$$\angle Y(j\omega)=\angle H(j\omega)+\angle W(j\omega) \quad (10)$$

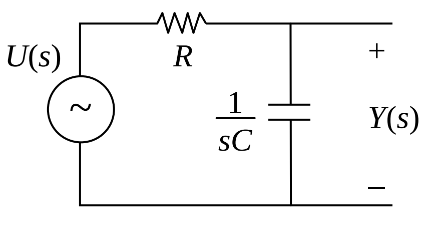
The frequency response given by (9) and (10) is often demonstrated by the Bode plot of $H(s)=\left.\frac{Q(s)}{P(s)}\right|_{s=j\omega}$.

There are three basic filters, named as lowpass, highpass and bandpass filters, to purposely let the low-, high- and band-frequency signals filter through.

The simplest lowpass filter is an RC circuit, as depicted in the following figure, and its transfer function is

$$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{1+sRC} \quad (11)$$

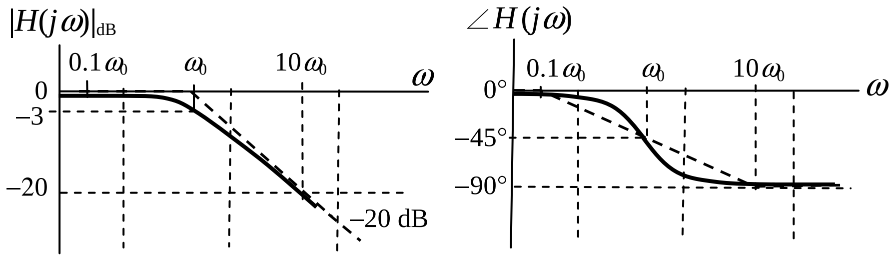
where *U*(s) and *Y*(s) are the input and output voltage signals.



Let $\omega_0=\frac{1}{RC}$, then the transfer function is reaaranged as $H(s)=\frac{1}{1+\frac{s}{\omega_0}}$ and its Fourier transform is

$$H(j\omega)=\frac{1}{1+j\dfrac{\omega}{\omega_0}}=\frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^2}}\angle\left(-tan^{-1}\frac{\omega}{\omega_0}\right) \quad (12)$$

The Bode plots of magnitude and phase are given in the following figures. From $|H(j\omega)|_{dB}$, it is clear that when $\omega > 10\omega_0$, the output $|Y(j\omega)|=|H(j\omega)|\ |U(j\omega)|$ will be highly decreased and less than 10% (or –20dB) of $|U(j\omega)|$. Since $\left|H(j\omega_0)\right|_{dB}\approx -3$ dB, we call the frequency ω_0 as 3dB point or cutoff frequency. Sometimes, ω_0 is also called the bandwidth of the lowpass filter.



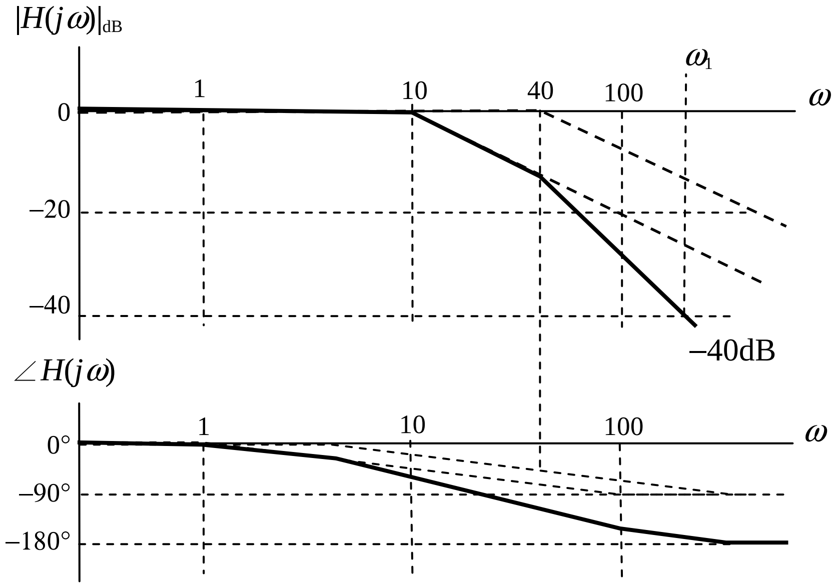
In addition the simplest *RC* lowpass filter, sometimes we will use a filter of higher order, such as a second order filter $H(s)=\dfrac{1}{1+as+bs^2}$ to keep the low-frequency component of the input signal and filter out the undesired high-frequency noise. An example is given below.

Example:

Consider a second order filter $H(s)=\dfrac{1}{\left(1+\dfrac{s}{10}\right)\left(1+\dfrac{s}{40}\right)}$ and draw the Bode plot. What is the range that $|Y(j\omega)|$ is less than 1% of $|U(j\omega)|$?

Sol:

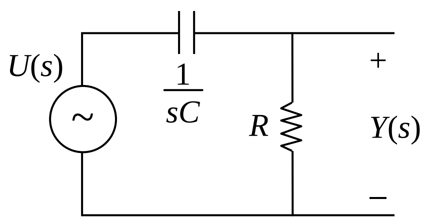
The fourier transform is $H(j\omega)=\dfrac{1}{\left(1+j\dfrac{\omega}{10}\right)\left(1+j\dfrac{\omega}{40}\right)}$, including two basic types $\left(1+j\dfrac{\omega}{10}\right)^{-1}$ and $\left(1+j\dfrac{\omega}{40}\right)^{-1}$, whose Bode plot is shown as below:



The range for $|Y(j\omega)|<0.01|U(j\omega)|$ is the range that $\left|H(j\omega)\right|_{\text{dB}}<-40\text{dB}$. From the Bode plot, it is the range of frequency higher than $\omega_1\approx 200$ rad/s. The frequency can be also calculated by

$$\left|H(j\omega_1)\right|=\frac{1}{\left|1+j\dfrac{\omega_1}{10}\right|\cdot\left|1+j\dfrac{\omega_1}{40}\right|}=\frac{1}{\sqrt{\left(1+\dfrac{\omega_1^2}{100}\right)\left(1+\dfrac{\omega_1^2}{1600}\right)}}=0.01,$$

which results in $\omega_1=198$ rad/s. That means if $\omega>198$ rad/s, then $|Y(j\omega)|$ is less than 1% of $|U(j\omega)|$.



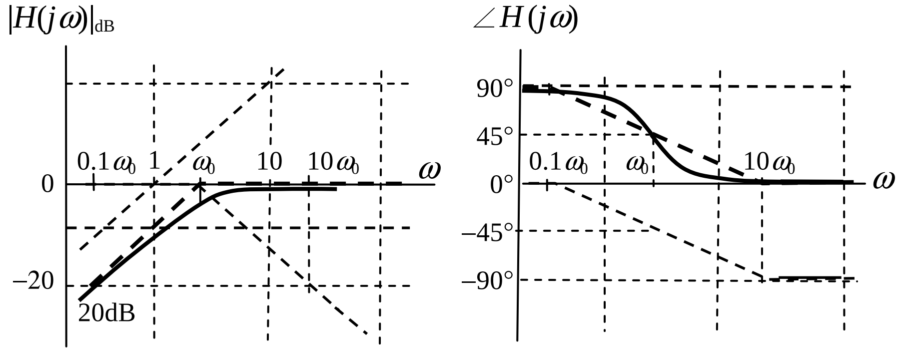
The *RC* circuit can be also used as a highpass filter if the output is the resistance voltage, shown on the right, whose transfer function is

$$H(s)=\frac{V_o(s)}{V_s(s)}=\frac{sRC}{1+sRC} \qquad (13)$$

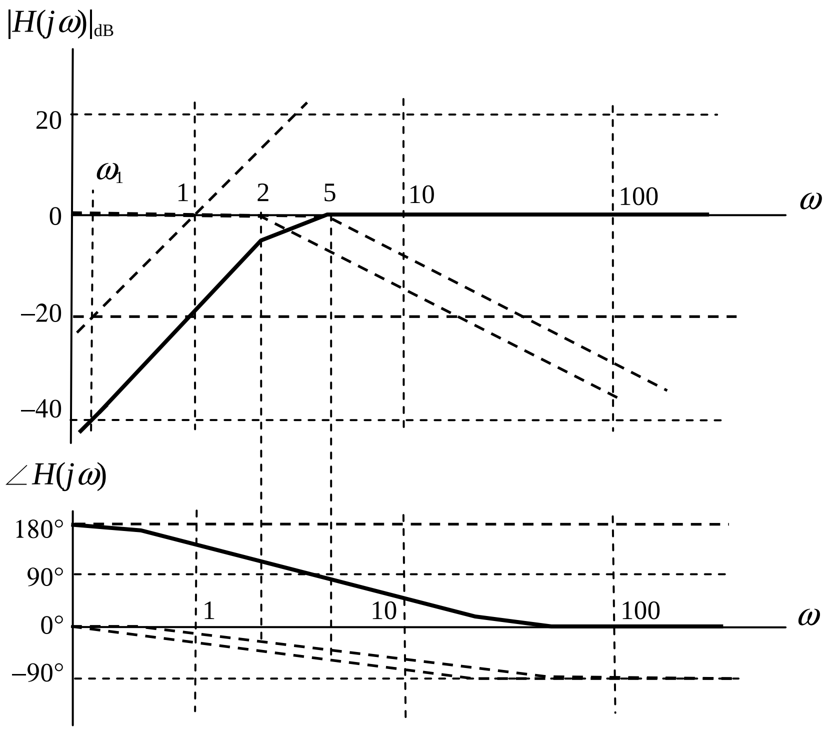
Let $\omega_0=\dfrac{1}{RC}$, then the Fourier transform is

$$H\left(j\omega\right)=\frac{j\omega/\omega_0}{1+j\dfrac{\omega}{\omega_0}}=\frac{\omega/\omega_0}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^2}}\angle\left(90^\circ-tan^{-1}\frac{\omega}{\omega_0}\right) \qquad (14)$$

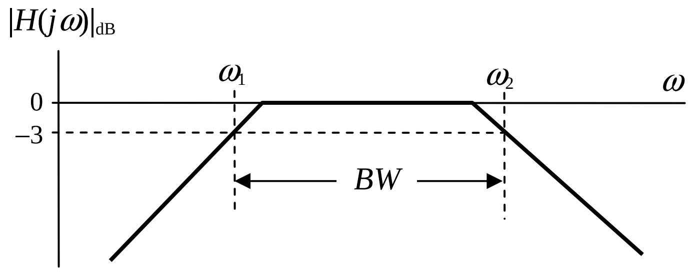
Assume 10>ω_0>1, then the Bode plot can be obtained as below.



In addition the simplest *RC* highpass filter, sometimes we will use a filter of higher order, such as a second order filter $H\left(s\right)=\dfrac{bs^2}{1+as+bs^2}$ to keep the high-frequency component of the input signal and filter out the undesired low-frequency noise. For example, the Bode plot of $H\left(s\right)=\dfrac{s^2/10}{\left(1+\dfrac{s}{2}\right)\left(1+\dfrac{s}{5}\right)}$ is



Since $\frac{1}{1+as+bs^2}$ is a lowpass filter and $\frac{bs^2}{1+as+bs^2}$ is a highpass filter, it implies that $\frac{as}{1+as+bs^2}$ is a bandpass filter whose Bode plot is shown as below:



Exercise:

Draw the Bode plot of the following system:

(A) $\dot{y}(t)+0.02y(t)=0.1u(t)$.

(B) $\ddot{y}(t)+0.07\dot{y}(t)+0.001y(t)=0.01\dot{u}(t)-0.05u(t)$.

(C) $\ddot{y}(t)+0.05\dot{y}(t)+0.02y(t)=0.1u(t)$.

[補充資料] Bode Plot

In signal analysis, the Bode plot of $H(j\omega)$ includes $|H(j\omega)|_{dB}$ and $\angle H(j\omega)$ respect to ω and scaled as $log_{10}\omega$. Some fundamental types of $Q(s)$ and $P(s)$ for $s=j\omega$ are listed below:

$$\begin{cases} Q_1(s)=K>0 \\ Q_2(s)=K<0 \\ Q_3(s)=s \\ Q_4(s)=1+\dfrac{s}{a}, \quad a>0 \\ Q_5(s)=1-\dfrac{s}{a}, \quad a>0 \\ Q_6(s)=\left(\dfrac{s}{\omega_n}\right)^2+2\xi\left(\dfrac{s}{\omega_n}\right)+1, \;\; 0\le\xi<1, \quad \omega_n>0 \\ Q_7(s)=\left(\dfrac{s}{\omega_n}\right)^2-2\xi\left(\dfrac{s}{\omega_n}\right)+1, \;\; 0\le\xi<1, \quad \omega_n>0 \end{cases} \tag{1}$$

$$\begin{cases} P_1(s)=1+\dfrac{s}{a}, \quad a>0 \\ P_2(s)=\left(\dfrac{s}{\omega_n}\right)^2+2\xi\left(\dfrac{s}{\omega_n}\right)+1, \;\; 0<\xi<1, \quad \omega_n>0 \end{cases} \tag{2}$$

For $Q_1(s)=K>0$ we have

$$|Q_1(j\omega)|_{dB}=20\,log_{10}K=M \quad \text{dB} \tag{3}$$

$$\angle Q_1(j\omega)=0° \tag{4}$$

where $M>0$ for $K>1$, $M=0$ for $K=1$ and $M<0$ for $0<K<1$. As for $Q_2(s)=K<0$, we have $|Q_2(j\omega)|_{dB}=|Q_1(j\omega)|_{dB}$ and $\angle Q_2(j\omega)=\pm180°$. Their Bode plots are shown in the next page where the ω-axis is scaled as $log_{10}\omega$.

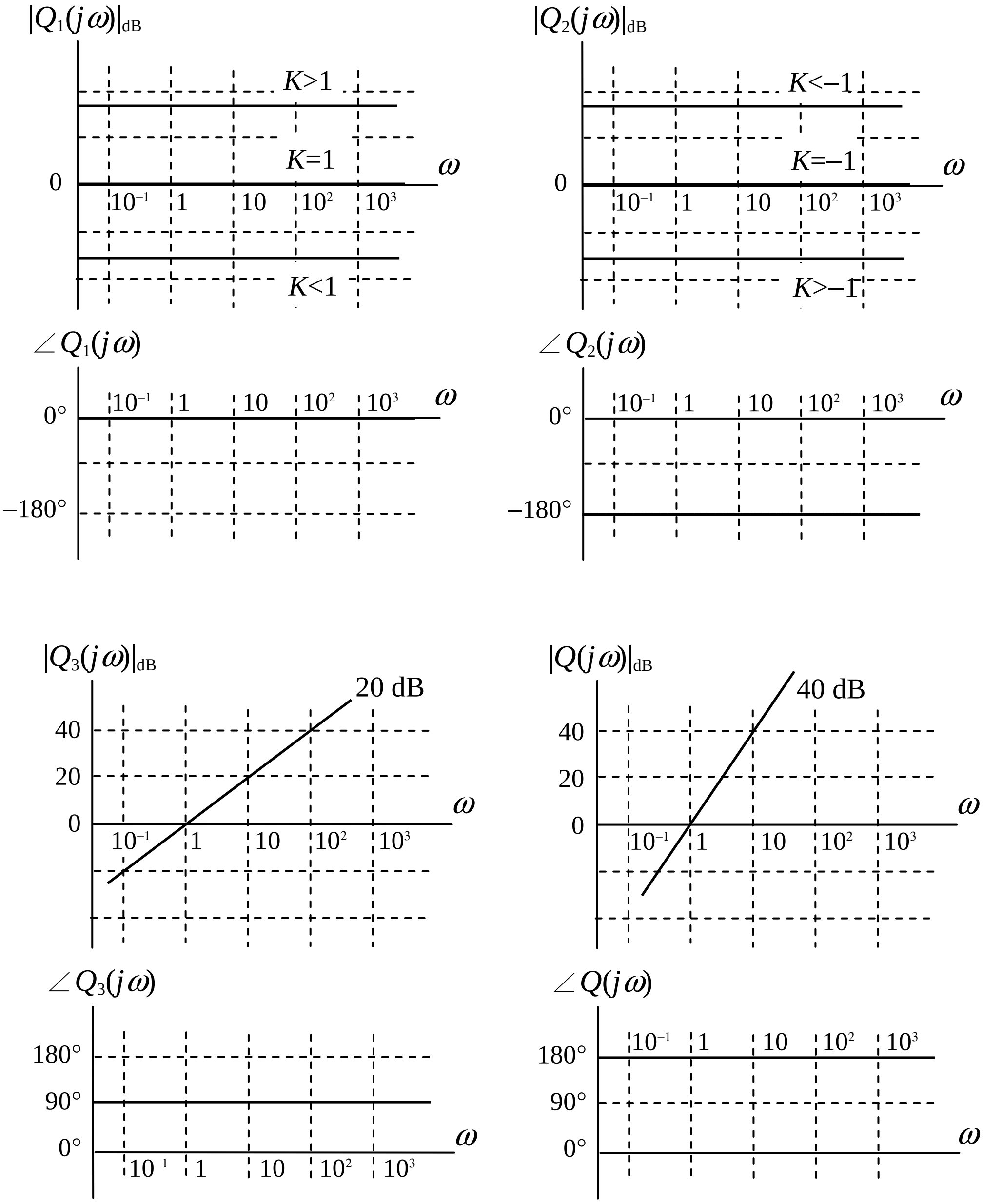
For $Q_3(s)=s$, which implies a zero is located at s=0, its Fourier transform is $Q_3(j\omega)=\omega\angle 90°$, i.e., the phase is 90° and the magnitude in dB is

$$|Q_3(j\omega)|_{dB}=20\,log_{10}\,\omega \quad \text{dB} \tag{5}$$

a straight line with slope 20dB/dec. If $Q(s)=s^n$, the Fourier transform is $Q(j\omega)=\omega^n\angle(n\cdot 90°)$ whose phase is $n\cdot$90° and magnitude in dB is

$$|Q(j\omega)|_{dB}=20\,log_{10}\,\omega^n=20n\,log_{10}\,\omega \quad \text{dB} \tag{6}$$

a straight line with slope 20ndB/dec. Note that their magnituide is 0dB at ω=1.

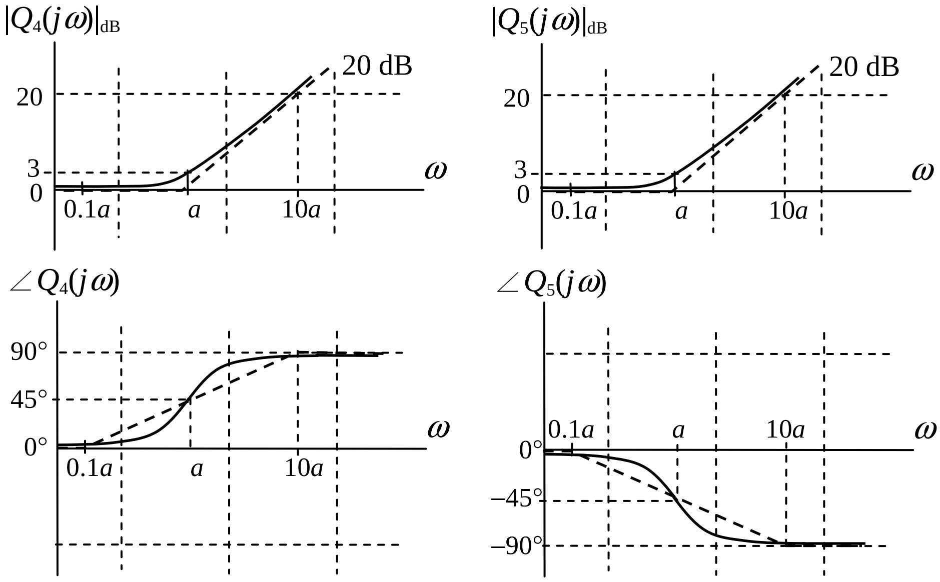


For $Q_4(s)=1+\frac{s}{a}$, $a>0$, i.e., the system contains a zero at $s=-a<0$ in the left-half complex plane, its Fourier transform is

$$Q_4(j\omega)=1+j\frac{\omega}{a}=\sqrt{1+\frac{\omega^2}{a^2}}\angle tan^{-1}\frac{\omega}{a} \tag{7}$$

whose phase is $\angle tan^{-1}\frac{\omega}{a}$ and magnitude in dB is

$$\left|Q_4(j\omega)\right|_{dB}=20\,log_{10}\sqrt{1+\frac{\omega^2}{a^2}}=10\,log_{10}\left(1+\frac{\omega^2}{a^2}\right)\quad \text{dB} \tag{8}$$



It is obtained that

$$\begin{cases} \angle Q_4\left(j0^+\right) = \angle tan^{-1}\dfrac{0^+}{a} \to 0^\circ \\ \left|Q_4\left(j0^+\right)\right|_{dB} = 10\,log_{10}\left(1+\dfrac{0^+}{a^2}\right) \to 0 \text{ dB} \end{cases}, \text{ for } \omega\to 0 \qquad (9)$$

$$\begin{cases} \angle Q_4\left(ja\right) = \angle tan^{-1}\dfrac{a}{a} = 45^\circ \\ \left|Q_4\left(ja\right)\right|_{dB} = 10\,log_{10}\left(1+\dfrac{a^2}{a^2}\right) = 3.01 \text{ dB} \end{cases}, \text{ for } \omega=a \qquad (10)$$

$$\begin{cases} \angle Q_4\left(j\infty\right) = \angle tan^{-1}\dfrac{\infty}{a} \to 90^\circ \\ \left|Q_4\left(j\omega\right)\right|_{dB,\omega\to\infty} = 10\,log_{10}\left(1+\dfrac{\omega^2}{a^2}\right)_{\omega\to\infty} \text{ dB} \end{cases}, \text{ for } \omega\to\infty \qquad (11)$$

Note that the magnitude in (11)can be further calculated as

$$\begin{aligned} \left|Q_4\left(j\omega\right)\right|_{dB,\omega\to\infty} &= 10\,log_{10}\left(\frac{\omega^2}{a^2}\right)_{\omega\to\infty} = 20\,log_{10}\left(\frac{\omega}{a}\right)_{\omega\to\infty} \\ &= 20\,log_{10}\,\omega\big|_{\omega\to\infty} - 20\,log_{10}\,a \end{aligned} \qquad (12)$$

For $Q_5(s) = 1-\dfrac{s}{a}$, a>0, i.e., i.e., the system contains a zero at $s\tilde{=}a>0$ in the right-half complex plane, its Fourier transform is

$$Q_5\left(j\omega\right) = 1 - j\frac{\omega}{a} = \sqrt{1+\frac{\omega^2}{a^2}}\angle\left(-tan^{-1}\frac{\omega}{a}\right) \qquad (13)$$

whose phase is $-\angle tan^{-1}\dfrac{\omega}{a}$ and magnitude in dB is the same as $\left|Q_4(j\omega)\right|_{dB}$.

For $Q_6(s)=\left(\dfrac{s}{\omega_n}\right)^2+2\xi\left(\dfrac{s}{\omega_n}\right)+1$, $1>\xi\geq 0$, $\omega_n>0$, i.e., the system has two conjudate complex zeros in the left-half complex plane, its Fourier transform is

$$\begin{aligned} Q_6(j\omega)&=\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\xi\frac{\omega}{\omega_n} \\ &=\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}\angle tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right) \end{aligned} \tag{14}$$

whose phase is $\angle tan^{-1}\left(\dfrac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right)$ and magnitude in dB is

$$\begin{aligned} \left|Q_6(j\omega)\right|_{dB}&=20\,log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2} \\ &=10\,log_{10}\left(1+\left(4\xi^2-2\right)\frac{\omega^2}{\omega_n^2}+\frac{\omega^4}{\omega_n^4}\right)\ \text{dB} \end{aligned} \tag{15}$$

it can be obtained that

$$\begin{cases} \angle Q_6(j0^+)=\angle tan^{-1}\dfrac{0^+}{\omega_n^2}\rightarrow 0^\circ \\ \left|Q_6(j0^+)\right|_{dB}=10\,log_{10}\left(1+\left(4\xi^2-2\right)\dfrac{0^+}{\omega_n^2}+\dfrac{0^+}{\omega_n^4}\right)\rightarrow 0\ \text{dB} \end{cases},\ \text{for } \omega\rightarrow 0 \tag{16}$$

$$\begin{cases} \angle Q_6(j\omega_n)=\angle tan^{-1}\infty=90^\circ \\ \left|Q_6(j\omega_n)\right|_{dB}=10\,log_{10}\left(4\xi^2\right)=6.02+20\,log_{10}\xi\ \text{dB} \end{cases},\ \text{for } \omega=\omega_n \tag{17}$$

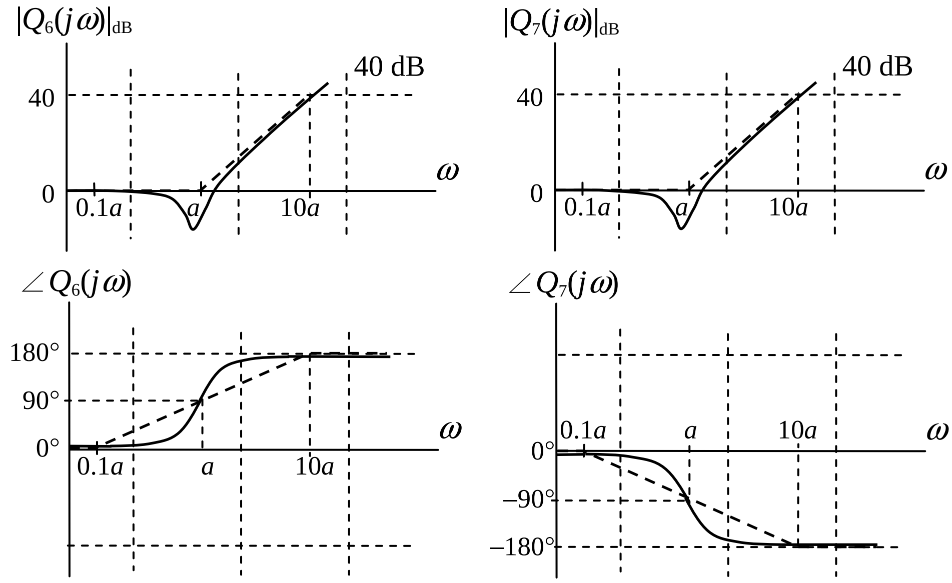
$$\begin{cases} \angle Q_6(j\infty)=\angle tan^{-1}\dfrac{\infty}{-\infty}\rightarrow 180^\circ \\ \left|Q_6(j\omega)\right|_{dB,\omega\rightarrow\infty}=10\,log_{10}\left(1+\left(4\xi^2-2\right)\dfrac{\omega^2}{\omega_n^2}+\dfrac{\omega^4}{\omega_n^4}\right)_{\omega\rightarrow\infty}\ \text{dB} \end{cases},$$

$$\text{for } \omega=\omega_n \tag{18}$$

Note that the magnitude in (28) can be approxmated as

$$\left|Q_6(j\omega)\right|_{dB,\omega\to\infty} = 10\,log_{10}\left(\frac{\omega^4}{\omega_n^4}\right)_{\omega\to\infty} = 40\,log_{10}\left(\frac{\omega}{\omega_n}\right)_{\omega\to\infty} = 40\,log_{10}\,\omega\Big|_{\omega\to\infty} - 40\,log_{10}\,\omega_n \quad (19)$$

with slope 40 dB/dec.

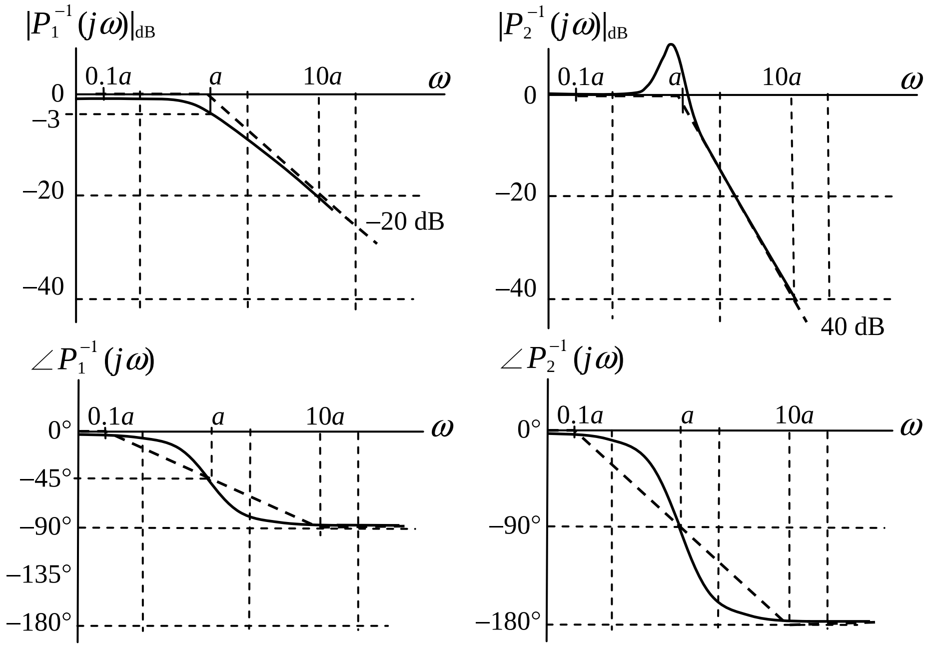


For $Q_7(s) = \left(\frac{s}{\omega_n}\right)^2 - 2\xi\left(\frac{s}{\omega_n}\right) + 1$, $1>\xi\geq 0$, $\omega_n>0$, the system has two conjugate zeros in the left half plane, its Fourier transform is

$$Q_7(j\omega) = \left(1-\frac{\omega^2}{\omega_n^2}\right) - j2\xi\frac{\omega}{\omega_n} = \sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}\,\angle tan^{-1}\left(\frac{-2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right) \quad (20)$$

whose phase is $\angle Q_7(j\omega) = -\angle Q_6(j\omega)$ amd magnitude in dB is the same as $\left|Q_6(j\omega)\right|_{dB}$.

For $P(s)$ in (12), where $P_1(s)=Q_4(s)$ and $P_2(s)=Q_6(s)$, the Bode plots of $P_1^{-1}(s)$ and $P_2^{-1}(s)$ are given in the following figure.



Example

Draw the Bode plot of $H(s)=\dfrac{50\,s(s-2)}{(s+10)(s^2+8s+25)}$.

<u>Sol:</u>

Re-express the transfer function as

$$H(s)=\frac{-0.4\,s\left(1-\dfrac{s}{2}\right)}{\left(1+\dfrac{s}{10}\right)\left(\left(\dfrac{s}{5}\right)^2+1.6\left(\dfrac{s}{5}\right)+1\right)}$$

and then we obtain the Fourier transform in the following

$$H(j\omega)=\frac{-0.4\,(j\omega)\left(1-j\dfrac{\omega}{2}\right)}{\left(1+j\dfrac{\omega}{10}\right)\left(\left(1-\dfrac{\omega^2}{5^2}\right)+j1.6\left(\dfrac{\omega}{5}\right)\right)}$$

which includes five fundamental types:

$$Q_2(j\omega) = -0.4, \quad Q_3(j\omega) = j\omega, \quad Q_5(j\omega) = 1 - j\frac{\omega}{2}$$

$$P_1(j\omega) = 1 + j\frac{\omega}{10}, \quad P_2(j\omega) = \left(1 - \frac{\omega^2}{5^2}\right) + j1.6\left(\frac{\omega}{5}\right)$$

represented by the dashed lines in the following Bode plot.