08. Laplace Transform

It has been introduced that the Fourier transform of a function *f*(*t*) can be expressed as

$$\Im\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{1}$$

and the inverse Fourier transform is

$$\Im^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \tag{2}$$

However, in real system any function should be started from a specific time $t=t_0$, called the initial time. Therefore, the Fourier transform is correspondingly given as

$$\Im\{f(t)\} = F(\omega) = \int_{t_0}^{\infty} f(t)e^{-j\omega t}dt \tag{3}$$

In system engineering, we often set the initial time as $t=0$, and thus (3) will be rewritten as

$$\Im\{f(t)\} = F(\omega) = \int_{0}^{\infty} f(t)e^{-j\omega t}dt \tag{4}$$

Unfortunately, when an impulse function is included in *f*(*t*), (4) is not a correct form to represent this case. Instead, we modify (4) into the following form

$$\Im\{f(t)\} = F(\omega) = \int_{0^-}^{\infty} f(t)e^{-j\omega t}dt \tag{5}$$

where $t=0^-$ is the time less but very close to $t=0$. With such a modification, any impulse function $\delta(t)$ occurred at $t=0$ will be calculated in (5).

In fact, the Fourier transform is useful for signal analysis, not for system analysis. To deal with the problem related to systems, some other mathematical tools are needed and one of them is the so-called Laplace transform.

Let *f*(*t*) be a function starting from $t=0$, i.e., $f(t)=0$ for t<0. Then, its Laplace transform is defined as

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{6}$$

where $s=\sigma+j\omega$ contains the real part $Re(s)=\sigma$ and the imaginary part $Im(s)=\omega$. It is noticed that if $s=j\omega$, along the imaginary axis, then (6) becomes

$$F(j\omega) = \int_{0^-}^{\infty} f(t)e^{-j\omega t}dt \tag{7}$$

Viewing from (5), it is known that the Fourier transform of *f*(*t*) is

$$\Im\{f(t)\} = F(\omega) = \int_{0^-}^{\infty} f(t)e^{-j\omega t}dt \qquad (8)$$

Obviously, it is the same as Laplace trasform with $s=j\omega$. It is notice that in (7) and (8) both $F(\omega)$ and $F(j\omega)$ have the same expression $\int_{0^-}^{\infty} f(t)e^{-j\omega t}dt$. Although that will cause a little confusion, they has been widely used in system engineering. What we have to know is that $F(\omega)$ represents the Fourier transform and $F(j\omega)$ is a special case of Laplace transform $F(s)$ for $s=j\omega$; that means both are equal to $\int_{0^-}^{\infty} f(t)e^{-j\omega t}dt$ and $F(\omega) \equiv F(j\omega)$.

According to the definition of Laplace transform in (6), it is required to know whether the integral is bounded or not. From (6) we have

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \int_{0^-}^{\infty} f(t)e^{-\sigma t - j\omega t}dt \qquad (9)$$

$$\le \int_{0^-}^{\infty} \left| f(t)e^{-\sigma t - j\omega t} \right| dt = \int_{0^-}^{\infty} \left| f(t)e^{-\sigma t} \right| \cdot \left| e^{-j\omega t} \right| dt = \int_{0^-}^{\infty} \left| f(t)e^{-\sigma t} \right| dt$$

where $\left| e^{-j\omega t} \right| = 1$. Under the condition that the real part $Re(s)=\sigma$ is large enough then the integral $\int_{0^-}^{\infty} \left| f(t)e^{-\sigma t} \right| dt$ is converged. Usually, we take the region as $Re(s)>-\alpha$ to ensure the convergence of $F(s)$ and call it the region of convergence, or ROC in short.

In addition to the convergence condition, it is also know that $F(s)$, the Laplace transform of $f(t)$, is unique; which implies $f(t)$ and $F(s)$ form a special mapping. If $f(t)$ is given, we can find its Laplace transform $F(s)$ by

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad (10)$$

and if F(s) is given, we can take the inverse Laplace transform to obtain $f(t)$ as below:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds \qquad (11)$$

which has been discussed in the course “Complex Variable.”

Let $\mathcal{L}\{f_1(t)\} = F_1(s)$ and $\mathcal{L}\{f_2(t)\} = F_2(s)$ and here we will discuss some important properties of the Laplace transform, which are listed as below:

(1) Linearity

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s) \qquad (12)$$

where a_1 and a_2 are constant.

(2) Time shift

$$\mathcal{L}\{f(t-a)\} = e^{-as}F(s) \tag{13}$$

where a is a constant and $f(t-a)=0$ for $t<a$.

(3) Frequency shift

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \tag{14}$$

where a is a constant.

(4) Time differentiation

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0^-) - \cdots - f^{(n-1)}(0^-) \tag{15}$$

where $f^{(n)}(t) \equiv \dfrac{d^n}{dt^n}$.

(5) Frequency differentiation

$$\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s) \tag{16}$$

where $F^{(n)}(s) \equiv \dfrac{d^n}{ds^n}$.

(6) Initial value theorem

$$f(0^+) = \lim_{s\to\infty} sF(s) \tag{17}$$

(7) Final value theorem

$$f(\infty) = \lim_{s\to 0} sF(s) \tag{18}$$

where $f(\infty)$ exists and must be a constant finite value.

(7) Convolution in time

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s) \tag{19}$$

<u>Pairs of Laplace transform:</u>

1	$\dfrac{1}{s}$	(20)
$e^{-\alpha t}$	$\dfrac{1}{s+\alpha}$	(21)
t^n	$\dfrac{n!}{s^{n+1}}$	(22)

$\cos \beta t$	$\dfrac{s}{s^2+\beta^2}$	(23)
$\sin \beta t$	$\dfrac{\beta}{s^2+\beta^2}$	(24)
$e^{-\alpha t}\cos \beta t$	$\dfrac{s+\alpha}{(s+\alpha)^2+\beta^2}$	(25)
$e^{-\alpha t}\sin \beta t$	$\dfrac{\beta}{(s+\alpha)^2+\beta^2}$	(26)

Exercise:

Find the Laplace transform $F(s)$ of the following $f(t)$ for $t \geq 0$:

(A) $f(t) = 2 + 3e^{-3t}$

(B) $f(t) = \cos 2t - e^{-t}$

(C) $f(t) = t^2 \sin 2t$

Exercise:

Find the inverse Laplace transform of the following $F(s)$ and determine $f(0)$ and $f(\infty)$:

(A) $F(s) = \dfrac{3s+1}{s(s^2+s+1)}$

(B) $F(s) = \dfrac{s^2+3}{s(s+1)(s+2)}$

(C) $F(s) = \dfrac{s^2+3}{s(s+1)^2}$

(D) $F(s) = \dfrac{s^2+3}{(s+1)^3}$

(E) $F(s) = \dfrac{s^2+3}{(s+2)(s^2+2s+5)}$

［補充資料］

皮爾-賽門·拉普拉斯

Pierre-Simon Laplace

1749 年 3 月 23 日- 1827 年 3 月 5 日

法國著名的天文學家和數學家

天體力學的集大成者

獨立於康德

提出科學的太陽系起源理論—星雲說

拉普拉斯於 1773 年進入法國科學學院，並且在 1799 年之後的二十多年間，陸陸續續出了五卷的《天體力學》，他也因這一部巨著被譽為法國的牛頓。據說，當拿破侖曾經問他為何在書中連一句上帝都不提，拉普拉斯回應說：「陛下，我不需要那個假設」。

《宇宙系統論》是拉普拉斯的另一部傑作，有別於康德的哲學角度，拉普拉斯在書中從數學與力學角度探討太陽系的起源理論—星雲說，因此人們常常把他們兩人的星雲說稱為「康德-拉普拉斯星雲說」。

除了天體力學與星雲理論外，拉普拉斯在機率方面也很有成就，他發現的許多相關理論大部分都發表在 1812 年出版的《關於概率的分析理論》一書中，其中包括貝氏定理(Bayes’ theorem)，當時他並不知該定理早已為貝斯所提出。

拉普拉斯最為一般學生所熟知的是拉普拉斯轉換(Laplace transformation)，但是最先使用拉普拉斯轉換的人是尤拉(Euler，1707-1783)，接著才是拉普拉斯，但是尤拉實在創作了太多的數學定理，所以數學界把他的一些次要創作，用下一位發現或應用的人來命名，拉普拉斯轉換即為一例。

拉普拉斯轉換之所以廣泛流通，還與英國的電機工程師奧利弗·亥維賽(Oliver Heaviside，1850-1925)有關，是他發現拉普拉斯轉換可以用來求解微分方程式。事實上亥維賽對電磁學也有很大的貢獻，他將麥克斯威爾方程組從原來的 20 條方程式減到 4 條微分方程。

在物理學中，拉普拉斯所發現的拉普拉斯方程式(Laplace Equation)是非常重要的公式，其中使用了運算子$\bigtriangledown^2$，為了紀念拉普拉斯的卓越貢獻，特別命名為拉普拉斯運算子(Laplace operator 或 Laplacian)。