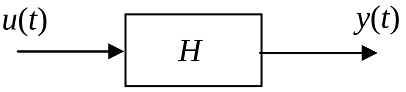
07. Linear Time-Invariant Systems



A system is depicted on the right which consists of input $u(t)$, output $y(t)$ and the system H itsself. In reality, if $y(t)$ is a result generated by the input $u(t)$, then it is impossible that $y(t)$ is received earlier than time that $u(t)$ sent into the system. For example, $y(t)=0$ for $t<0$ if $u(t)$ starts to excite the system at $t=0$. This kind of system is called the causal system. A system operating in real time must be a causal system. However, it is not necessary for a system which is not processed in real time. For example, the process of an image taken at a specific time is not causal.

In system engineering, the most common system is the linear time-invariant (LTI) system, which satisfies the superposition and is independent to the time shifting. For simplicity in explanation, the system operation is often described as below:

$$y(t) = H[u(t)] \tag{1}$$

which implies the output $y(t)$ is a function of $u(t)$. Let $y_1(t)$ and $y_2(t)$ be the outputs of $u_1(t)$ and $u_2(t)$, respectively, i.e.,

$$y_1(t) = H[u_1(t)] \tag{2}$$

$$y_2(t) = H[u_2(t)] \tag{3}$$

If the system satisfies

$$\begin{aligned} H[\alpha_1 u_1(t) + \alpha_2 u_2(t)] &= \alpha_1 H[u_1(t)] + \alpha_2 H[u_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned} \tag{4}$$

for any constant α_1 and α_2, then the system is a linear system. The property in (4) is the so-called superposition. If the number of inputs is more than two, (4) can be further extended as

$$H\left[\sum_{i=1}^{m} \alpha_i u_i(t)\right] = \sum_{i=1}^{m} \alpha_i H[u_i(t)] = \sum_{i=1}^{m} \alpha_i y_i(t) \tag{5}$$

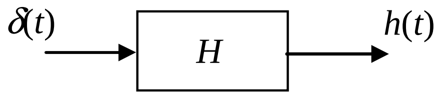
where $y_i(t) = H[u_i(t)]$, $i=1,2,\ldots,m$.

As for the time-invariant property, it states that the output of a system is shifted in time by τ when the input is shifted by τ, i.e.,

$$y(t-\tau) = H[u(t-\tau)] \tag{6}$$

Clearly, the operation of a time-invariant system does not depend on the time it is excited.

To sum up, an LTI system must satisfy the superposition (5) and the time-invariant property (6).



It is known that an LTI system is often represented by its impulse response $h(t)$, i.e., $y(t)=h(t)$ for $u(t)=\delta(t)$, or

$$h(t)=H[\delta(t)] \tag{7}$$

Before introducing the relation of $u(t)$ and $y(t)$, let's express $u(t)$ in terms of $\delta(t)$ as

$$u(t)=\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau \tag{8}$$

which can be further changed into the following expression:

$$u(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty} u(k\,\Delta\tau)\delta(t-k\,\Delta\tau)\Delta\tau \tag{9}$$

or

$$u(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty} \alpha_k\, u_k(t) \tag{10}$$

Note that the coefficients $\alpha_k = u(k\,\Delta\tau)\Delta\tau$ do not depend on t and $u_k(t)=\delta(t-k\,\Delta\tau)$ is the k-th component of $u(t)$. Clearly, $u(t)$ has been decomposed into infinite impulse functions.

Now, let's show that for an LTI system the output $y(t)$ equals the convolution of the input $u(t)$ and the impulse response $h(t)$, i.e.,

$$y(t)=h(t)*u(t)=u(t)*h(t) \tag{11}$$

where the convolution is commutative. First, from (1), (8) and (10) we have

$$y(t)=H[u(t)]=H\left[\int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau\right]=\lim_{\Delta\tau\to 0} H\left[\sum_{k=-\infty}^{\infty}\alpha_k\, u_k(t)\right] \tag{12}$$

Since the system is linear and thus

$$y(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty} \alpha_k\, H[u_k(t)] \tag{13}$$

with $\alpha_k = u(k\,\Delta\tau)\Delta\tau$ and $u_k(t)=\delta(t-k\,\Delta\tau)$. Hence,

$$y(t)=\lim_{\Delta\tau\to 0}\sum_{k=-\infty}^{\infty} u(k\,\Delta\tau)H[\delta(t-k\,\Delta\tau)]\Delta\tau \tag{14}$$

which leads to the integral form as

$$y(t)=\int_{-\infty}^{\infty} u(\tau)H[\delta(t-\tau)]\,d\tau \tag{15}$$

Further applying the time-invariant property (6), we have

$$h(t-\tau)=H[\delta(t-\tau)] \tag{16}$$

where $h(t)$ is the impulse response given in (7). Now, Substituting (16) into (15) yields

$$y(t)=\int_{-\infty}^{\infty}u(\tau)h(t-\tau)d\tau=u(t)*h(t)=h(t)*u(t) \tag{17}$$

which proves (11).

According to the property of convolution in time of Fourier transform, if $Y(\omega)$, $H(\omega)$ and $U(\omega)$ are the Fourier transforms of $y(t)$, $h(t)$ and $u(t)$, then

$$Y(\omega)=H(\omega)U(\omega) \tag{18}$$

which has been widely used in filter design.

Consider the case that $u(t)$ starts to excite the system at t=0, i.e., $u(t)$=0 for t<0. Since the impulse response $h(t)$ is a special output related to the input $\delta(t)$, we also know that $h(t)$=0 for t<0 based on the causality. As a result, the convolution in (17) is rewritten as

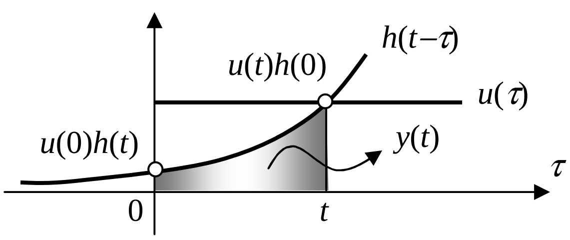
$$y(t)=\int_{0}^{t}u(\tau)h(t-\tau)d\tau \tag{19}$$

which is a kind of running integral. Clearly, the output $y(t)$ only depends on the input $u(\tau)$ for $0\le\tau\le t$. Indeed, (19) represents a causal system. Most importantly, y(t) requires the history of $u(t)$ and therefore the system must be possessed of 'memory' components to record the past of $u(t)$.

Later, we will learn that the impulse response of an LTI system often consists of decreased exponential function, such as e^{-at} where a>0. To demonstrate the calculation of convolution, let's use the the following example with u(t)=1 and h(t)=e^{-2t} for $t\ge 0$. Hence, the output will be obtained as

$$y(t)=\int_{0}^{t}u(\tau)h(t-\tau)d\tau=\int_{0}^{t}e^{-2(t-\tau)}\,d\tau=0.5\left(1-e^{-2t}\right) \tag{20}$$

which is depicted in the following figure.



To explain the convolution in the above figure, according to (14) we change theoutput in (20) as

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{k=0}^{n} u(k\,\Delta\tau) h(t - k\,\Delta\tau)\Delta\tau \qquad (21)$$

where $t=n\Delta\tau$. That means

$$\begin{aligned} y(t) &= u(0)h(n\Delta\tau)\Delta\tau + \cdots + u(k\Delta\tau)h((n-k)\Delta\tau)\Delta\tau \\ &\quad + u((k+1)\Delta\tau)h((n-(k+1))\Delta\tau)\Delta\tau + \cdots + u(n\Delta\tau)h(0)\Delta\tau \end{aligned} \qquad (22)$$

i.e., $y(t)$ is composed of terms $u(k\Delta\tau)h((n-k)\Delta\tau)\Delta\tau$. Because $h((n-k)\Delta\tau)$ is less than $h((n-(k+1)\Delta\tau)$, we can conclud that $y(t)$ is more affected by $u((k+1)\Delta\tau)$ than $u(k\Delta\tau)$. In other words, the effect caused by the past $u(t)$ is faded away as time advances on.

Exercise:

Find the convolution of the following functions:

(A) $f(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}$ and $g(t) = \begin{cases} e^{-3t}, & t \ge 0 \\ 0, & \text{elsewhere} \end{cases}$

(B) $f(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}$ and $g(t) = \begin{cases} sin\frac{1}{2}t, & t \ge 0 \\ 0, & \text{elsewhwere} \end{cases}$