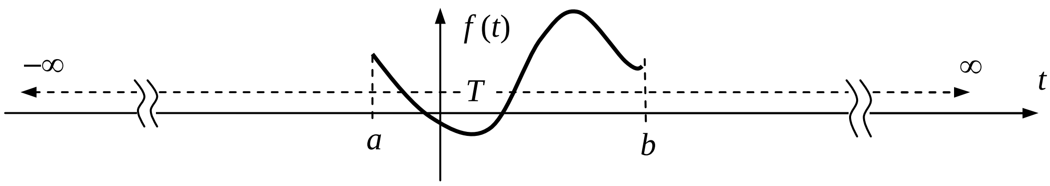
06. Fourier Transform

Now, one question is raised: Is that possible to find a unique expression for any functions, not just periodic functions, satisfying Dirichlet conditions? The answer is “Yes,” but why?



Any finite duration function $f(t)$, depicted in the above figure, can be treated as a periodic function with period $T=\infty$. Then, according to Fourier series we have

$$f(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\ . \tag{1}$$

where

$$c_k=\frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}\,dt \tag{2}$$

Since $T=\dfrac{2\pi}{\omega_0}$, substituting (2) into (1) yields

$$f(t)=\sum_{\substack{k=-\infty \\ T\to\infty}}^{\infty}\left(\frac{\omega_0}{2\pi}\int_{-T/2}^{T/2} f(\tau)e^{-jk\omega_0\tau}\,d\tau\right)e^{jk\omega_0 t}\ . \tag{3}$$

Under the assumption $T\to\infty$, let $\omega_0=\Delta\omega\to 0$ and $k\omega_0=k\Delta\omega\to\omega$ where ω is a continuous variable. Therefore, (3) can be changed into

$$f(t)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\,d\tau\right)e^{j\omega t}\Delta\omega. \tag{4}$$

Further taking $\Delta\omega$ as $d\omega$, (4) can be written as an integral form as below

$$f(t)=\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\,d\tau\right)e^{j\omega t}d\omega. \tag{5}$$

If we define

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\ . \tag{6}$$

then

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega\ . \tag{7}$$

Usually, we call $F(\omega)$ in (6) is the Fourier transform of $f(t)$, which is symbolized as

$$\Im\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \quad (8)$$

and $f(t)$ is the inverse Fourier transform of $F(\omega)$, which is symbolized as

$$\Im^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega \quad (9)$$

Besides, from (7), we know that $f(t)$ is formed by an infinite number of terms $\frac{1}{2\pi}F(\omega)$ where ω is the continuoius frequency.

The fourier transform $F(\omega)$ in (6) is a complex number and it can be expressed by the following form:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = |F(\omega)|\, e^{j\phi(\omega)}. \quad (10)$$

It is obvious that

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t}\, dt = F^{*}(\omega). \quad (11)$$

which leads to

$$|F(-\omega)|\, e^{j\phi(-\omega)} = |F(\omega)|\, e^{-j\phi(\omega)}. \quad (12)$$

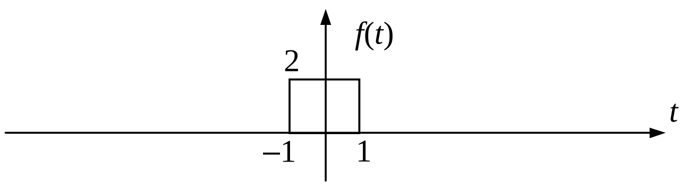
In other words,

$$|F(-\omega)| = |F(\omega)| \ . \quad (13)$$

$$\phi(-\omega) = -\phi(\omega). \quad (14)$$

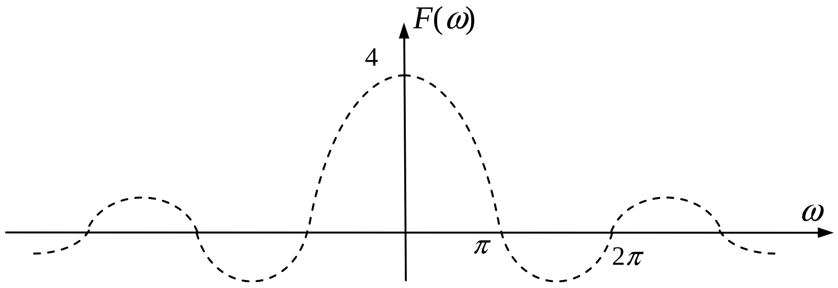
which implies $F(\omega)$ is an even function and $\phi(\omega)$ is an odd function.

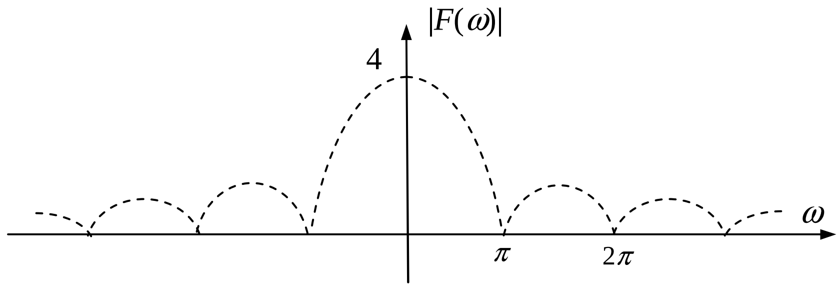
Example

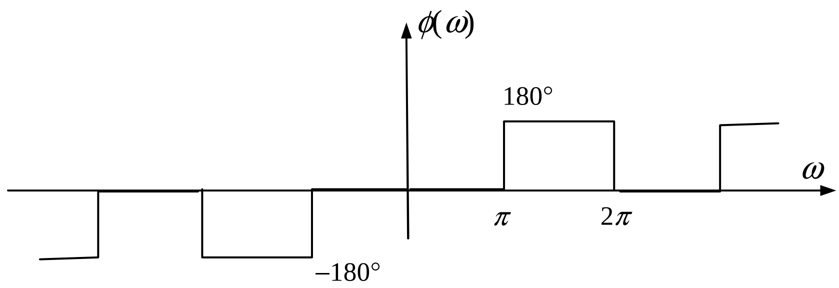


The fourier transform is given as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \int_{-1}^{1} 2e^{-j\omega t}\, dt = \frac{4\sin\omega}{\omega} = 4\,\mathrm{sinc}(\omega)$$







It has been shown that a periodic function can be represented by the Fourier transform as

$$f(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\ . \tag{15}$$

Then, what is its Fourier transform? Based on the definition (6), we have

$$\Im\{f(t)\}=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right)e^{-j\omega t}\,dt \tag{16}$$

$$=\sum_{k=-\infty}^{\infty} c_k\left(\int_{-\infty}^{\infty} e^{-j(\omega-k\omega_0)t}dt\right)$$

Hence, it is required to find the expression of $\int_{-\infty}^{\infty} e^{-j(\omega-k\omega_0)t}dt$. Let's check the following inverse Fourier transform

$$\Im^{-1}\{\delta(\omega-k\omega_0)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-k\omega_0)e^{j\omega t}\,d\,\omega=\frac{1}{2\pi}e^{jk\omega_0 t} \tag{17}$$

which implies

$$\Im\left\{e^{jk\omega_0 t}\right\}=\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t}\, d\,\omega=\int_{-\infty}^{\infty} e^{-j(\omega-k\omega_0)t}\, d\,\omega \\ =2\pi\delta(\omega-k\omega_0) \tag{18}$$

Therefore, (16) can be rewritten as

$$\Im\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\}=2\pi\sum_{k=-\infty}^{\infty} c_k \delta(\omega-k\omega_0) \tag{19}$$

which is the frequency domain description of a periodic function. It consists of a sequence of impulses with weight $2\pi c_k$ and presents a discrete frequency spectrum.

Let $F(\omega)$ and $G(\omega)$ be the Fourier transform of $f(t)$ and $g(t)$, respectively. Some important properties often used in Fourier transform are listed below:

(1) Linearity

$$\Im\{a\cdot f(t)+b\cdot g(t)\}=a\cdot F(\omega)+b\cdot G(\omega) \tag{20}$$

(2) Time-shifting

$$\Im\{f(t-t_0)\}=e^{-j\omega t_0}\, F(\omega) \tag{21}$$

(3) Frequency-shifting

$$\Im\left\{e^{j\omega_0 t} f(t)\right\}=F(\omega-\omega_0) \tag{22}$$

(4) Time compression and expansion

$$\Im\{f(at)\}=\frac{1}{a}F\left(\frac{\omega}{a}\right) \tag{23}$$

(5) Convolution in time

$$\Im\left\{\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau\right\}=F(\omega)G(\omega) \tag{24}$$

(6) Multiplication in time

$$\Im\{f(t)g(t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega-\Omega)G(\Omega)d\Omega \tag{25}$$

Some other properties will be also discussed. Let TD be the time duration of function $f(t)$ and BW be the frequency bandwidth of $F(\omega)$, then TD is approximately proportional to $1/BW$, i.e.,

$$TD\sim\frac{1}{BW} \tag{26}$$

The instantaneous power of a function $f(t)$ is often expressed as $f^2(t)$ or, if $f(t)$ is a complex number, $f(t)\,f^*(t)$. Then, the total energy is given as

$$E=\int_{-\infty}^{\infty} f(t)\,f^*(t)d\,t=\int_{-\infty}^{\infty}\left|f(t)\right|^2(t)d\,t \tag{27}$$

no matter *f*(*t*) is real or complex. From (7), we have

$$E = \int_{-\infty}^{\infty} f(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}\,d\omega\right)dt \tag{28}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\omega$$

Therefore,

$$\int_{-\infty}^{\infty}\left|f(t)\right|^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|^2 d\omega \tag{29}$$

which is called the parseval theorem.

Exercise:

Find the magnitude and phase spectra of the following periodic functions:

(A) $f(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & \text{elsewhere} \end{cases}$

(B) $g(t) = \begin{cases} sin\frac{1}{2}t, & -\pi < t < \pi \\ 0, & \text{elsewhwere} \end{cases}$