05. Frequency Spectra of Fourier Series

It has been pointed out that we can use the Fourier series coefficients to represent a periodic function which satisfies the Dirichlet conditions. The Fourier series of $f(t)$ with period T is expressed as

$$f(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k \cos k\omega_0 t + B_k \sin k\omega_0 t\right) \quad \text{for } t \in (-\infty, \infty) \tag{1}$$

where $\omega_0 = \dfrac{2\pi}{T}$ and

$$A_0 = \frac{1}{T}\int_a^b f(t)dt = \frac{1}{T}\int_T f(t)dt \tag{2}$$

$$A_k = \frac{2}{T}\int_T f(t)\cos k\omega_0 t\, dt \tag{3}$$

$$B_k = \frac{2}{T}\int_T f(t)\sin k\omega_0 t\, dt \tag{4}$$

In system engineering, the Fourier series (1) in trigonometric form is often changed into the complex form by the use of

$$\cos k\omega_0 t = \frac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) \tag{5}$$

$$\sin k\omega_0 t = \frac{1}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right). \tag{6}$$

Substituting (5) and (6) into (1) yields

$$f(t) = A_0 + \sum_{k=1}^{\infty}\left(\frac{A_k}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \frac{B_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)\right). \tag{7}$$

Further rearranging (7) results in

$$f(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \tag{8}$$

where

$$c_0 = A_0 \tag{9}$$

$$c_k = \frac{1}{2}\left(A_k - jB_k\right) = \left|c_k\right| e^{j\phi_k}, \qquad k \geq 1 \tag{10}$$

$$c_{-k} = \frac{1}{2}\left(A_k + jB_k\right) = \left|c_k\right| e^{-j\phi_k} = \left|c_{-k}\right| e^{j\phi_{-k}}, \qquad k \geq 1 \tag{11}$$

From (10), for $k \geq 1$ we have

$$c_k = \frac{1}{2}(A_k - jB_k) = \frac{1}{T}\int_T f(t)(\cos k\omega_0 t - j\sin k\omega_0 t)dt$$
$$= \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}\,dt = |c_k|\,e^{j\phi_k} \quad (12)$$

where

$$|c_k| = \frac{1}{2}\sqrt{A_k^2 + B_k^2} \quad \text{and} \quad \phi_k = \tan^{-1}\left(\frac{-B_k}{A_k}\right) \quad (13)$$

From (10) and (11), it is clear that

$$c_{-k} = c_k^* \quad (14)$$

and

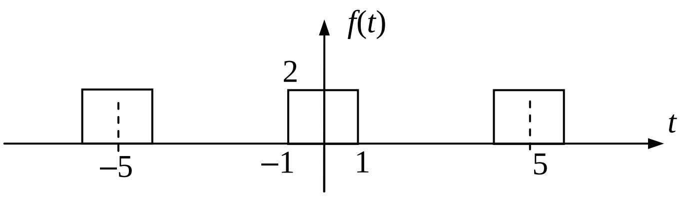
$$|c_{-k}| = |c_k| \quad (15)$$

$$\phi_{-k} = -\phi_k \quad (16)$$

which implies the amplitude $|c_k|$ is an even function and ϕ_k is an odd function with respect to k.

According to (8), the amplitude $|c_k|$ and the phase ϕ_k are respectively related to the frequency $\omega = k\omega_0$. If we draw the function $|c_k|$ with respect to $\omega = k\omega_0$ then we have the amplitude spectrum of $f(t)$. Similarly, if we draw the function ϕ_k with respect to $\omega = k\omega_0$ then we have the phase spectrum of $f(t)$. Both spectra are unique and can be used to represent the function $f(t)$. Since $|c_k|$ and ϕ_k only exist at integer k, their spectra are called discrete frequency spectrum.

Example



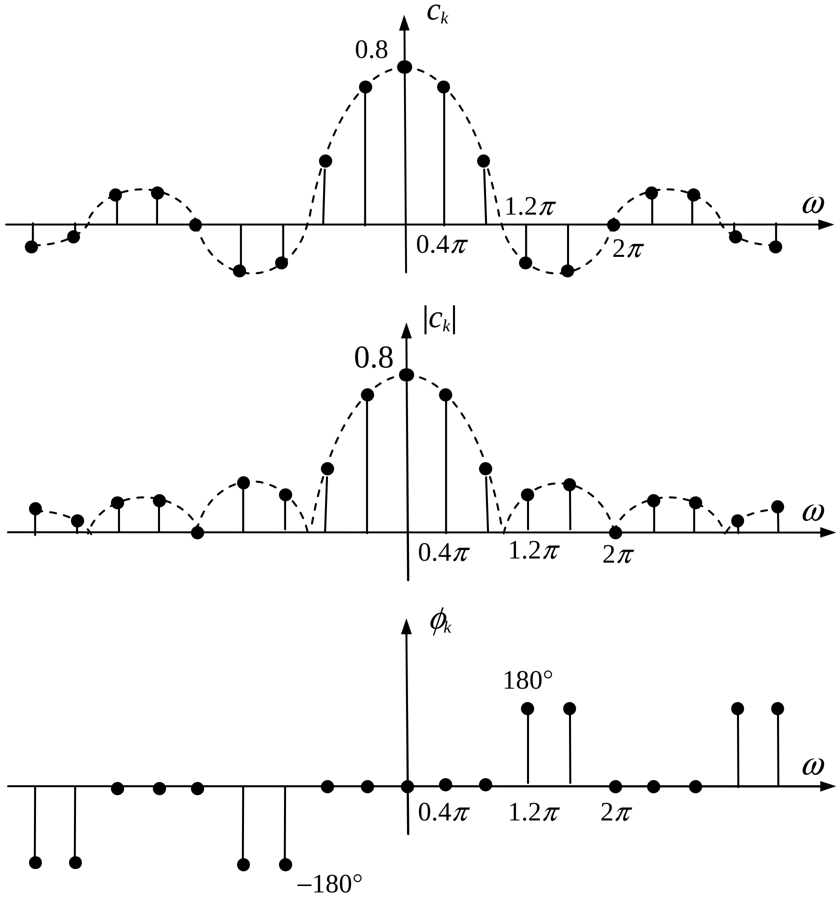
Since T=5, we have $\omega_0 = 2\pi/T = 0.4\pi$. From (12), it can be attained that

$$c_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}\,dt = \frac{1}{5}\int_{-1}^{1}\left(2e^{-j0.4k\pi}\right)dt$$
$$= \frac{1}{-jk\pi}e^{-j0.4k\pi}\Big|_{-1}^{1} = \frac{2}{k\pi}\sin(0.4k\pi) = 0.8\left(\frac{\sin(0.4k\pi)}{0.4k\pi}\right)$$
$$= 0.8\,\text{sinc}(0.4k\pi)$$

corresponding to $\omega = k\omega_0 = 0.4k\pi$. Clearly, c_k is a real number, i.e., the amplitude

$$|c_k| = 0.8\left|\frac{sin(0.4k\pi)}{0.4k\pi}\right| = 0.8\left|sinc(0.4k\pi)\right|$$

and the phase $\phi_k = 0$ or ±180°.



The power of a function $f(t)$ is often defined by $f^2(t)$ and thus the average power of a periodic function is often described by mean-square value, given as

$$P = \frac{1}{T}\int_T f^2(t)dt \tag{17}$$

Based on the Fourier series, we have

$$P = \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right)dt \tag{18}$$

$$= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \sum_{m=-\infty}^{\infty} c_{-m} e^{-jm\omega_0 t}\right)dt$$

$$= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k c_{-m} e^{j(k-m)\omega_0 t}\right)dt$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{T} \int_T c_k c_{-m} e^{j(k-m)\omega_0 t} dt$$

It is known that

$$\int_T c_k c_{-m} e^{j(k-m)\omega_0 t} dt \ = 0 \qquad \text{for } k \neq m \tag{19}$$

Hence, (18) can be rewritten as

$$P = \sum_{k=-\infty}^{\infty} \frac{1}{T} \int_T c_k c_{-k} dt = \sum_{k=-\infty}^{\infty} c_k c_{-k} \ = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{20}$$

From (17) and (20), we obtain

$$P = \frac{1}{T} \int_T f^2(t) dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{21}$$

which is called the Parseval's theorem concerning the average power of a periodic function.

Exercise:

Find the magnitude and phase spectra of the following periodic functions:

(A) $f(t+4) = f(t)$ and $f(t) = f_T(t)$ for $0 < t < 4$

where $f_T(t) = \begin{cases} 1, & 0 < t < 2 \\ 0, & 2 < t < 4 \end{cases}$

(B) $g(t+2\pi) = g(t)$ and $g(t) = g_T(t)$ for $0 < t < 2\pi$

where $g_T(t) = sin\frac{1}{2}t$