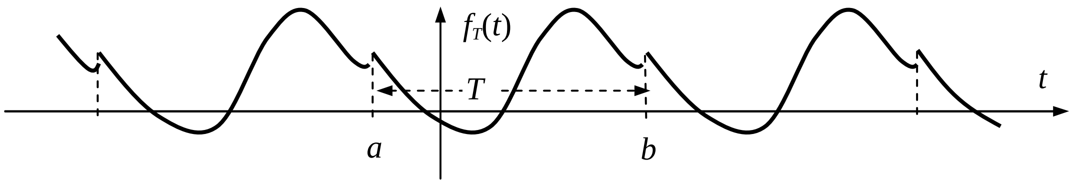
04. Fourier Series

The Fourier series was propsed by French mathematician Jean Baptiste Joseph Fourier(1768-1830) and mainly applied to periodic functions.



The above figure is a periodic function $f_T(t)$, $-\infty<t<\infty$, with period $T=b-a$. Based on Fourier series, the periodic function can be expressed as

$$f_T(t) = A_0 + \sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T} + B_k sin\frac{2k\pi t}{T}\right) \quad \text{for } t\in(-\infty,\infty) \qquad (1)$$

where

$$A_0 = \frac{1}{T}\int_a^b f_T(t)dt = \frac{1}{T}\int_T f_T(t)dt \qquad (2)$$

$$A_k = \frac{2}{T}\int_T f_T(t)cos\frac{2k\pi t}{T}dt \qquad (3)$$

$$B_k = \frac{2}{T}\int_T f_T(t)sin\frac{2k\pi t}{T}dt \qquad (4)$$

It has been proved that Fourier series could represent a periodic function which satisfies the following Dirichlet conditions:

C1- In one period, the number of discontinuous points is finite.

C2- In one period, the number of maximum and minimum points is finite.

C3- In one period, the function is absolutely integrable.

It is interesting to point out that although all the infinite number of sinusoidal functions in (1) are continuous, their sum may be discontinuous just like the function $f_T(t)$ depicted in the figure, which satisfies C1~C3.

Next, let's derive the expression of the coefficients in (2), (3) and (4). For the coefficient A_0, we take the integration

$$\begin{aligned}\int_a^b f_T(t)dt &= \int_a^b\left[A_0 + \sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T} + B_k sin\frac{2k\pi t}{T}\right)\right]dt \\ &= \int_a^b A_0 dt + \sum_{k=1}^{\infty} A_k\int_a^b cos\frac{2k\pi t}{T}dt + \sum_{k=1}^{\infty} B_k\int_a^b sin\frac{2k\pi t}{T}dt\end{aligned} \qquad (5)$$

Since $\int_a^b cos\frac{2k\pi t}{T}dt=0$ and $\int_a^b sin\frac{2k\pi t}{T}dt=0$, we have

$$\int_a^b f_T(t)dt=\int_a^b A_0 dt=A_0(b-a)=A_0T \tag{6}$$

which results in

$$A_0=\frac{1}{T}\int_a^b f_T(t)dt=\frac{1}{T}\int_T f_T(t)dt \tag{7}$$

where $\int_a^b$ is denoted as $\int_T$ to show integral duration is T. Clearly, the coefficient A_0 is the mean value of $f_T(t)$ for $t\in(a,b)$.

For the the coefficients A_k, we take the following integration from a to b, which is expressed as

$$\begin{aligned}\int_a^b f_T(t)cos\frac{2m\pi t}{T}dt&=\int_T f_T(t)cos\frac{2m\pi t}{T}dt\\&=\int_T\left[A_0+\sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T}+B_k sin\frac{2k\pi t}{T}\right)\right]cos\frac{2m\pi t}{T}dt\\&=\sum_{k=1}^{\infty}A_k\int_T cos\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt+\sum_{k=1}^{\infty}B_k\int_T sin\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt\end{aligned} \tag{8}$$

where $m\geq 1$, $m\in N$ and $\int_T A_0 cos\frac{2m\pi t}{T}dt=0$. It is known that

$$\int_T cos\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt=\frac{1}{2}\int_T\left(cos\frac{2(k+m)\pi t}{T}+cos\frac{2(k-m)\pi t}{T}\right)dt \tag{9}$$

$$\int_T sin\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt=\frac{1}{2}\int_T\left(sin\frac{2(k+m)\pi t}{T}+sin\frac{2(k-m)\pi t}{T}\right)dt \tag{10}$$

where $\int_T cos\frac{2(k+m)\pi t}{T}dt=0$, $\int_T sin\frac{2(k+m)\pi t}{T}dt=0$, and $\int_T sin\frac{2(k-m)\pi t}{T}dt=0$ for k, $m\in N$. Hence, (9) and (10) can be further written into

$$\int_T cos\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt=\frac{1}{2}\int_T cos\frac{2(k-m)\pi t}{T}dt=\begin{cases}T/2, & k=m\\ 0, & k\neq m\end{cases} \tag{11}$$

$$\int_T sin\frac{2k\pi t}{T}cos\frac{2m\pi t}{T}dt=0 \tag{12}$$

Therefore, the two summations in (8) can be evaluated as :below:

$$\sum_{k=1}^{\infty} A_k \int_T cos\frac{2k\pi t}{T} cos\frac{2m\pi t}{T} dt = A_m T/2 \qquad (13)$$

$$\sum_{k=1}^{\infty} B_k \int_T sin\frac{2k\pi t}{T} cos\frac{2m\pi t}{T} dt = 0 \qquad (14)$$

The truth of (13) is a result of (11), whose integration does not vanish only for $k=m$. As for (14), it can be easily derived from (12), whose integration is zero for all $k\in N$. Hence, from (8) we have

$$\int_T f_T(t) cos\frac{2m\pi t}{T} dt = A_m T/2 \qquad \text{for } m\in N \qquad (15)$$

which is the same as

$$\int_T f_T(t) cos\frac{2k\pi t}{T} dt = A_k T/2 \qquad \text{for } k\in N \qquad (16)$$

Then, the coefficient A_k in (3) is evaluated as

$$A_k = \frac{2}{T}\int_T f_T(t) cos\frac{2k\pi t}{T} dt \qquad \text{for } k\in N \qquad (17)$$

Similarly, the coefficients B_k can be found from the following integration

$$\begin{aligned}\int_a^b f_T(t) sin\frac{2m\pi t}{T} dt &= \int_T f_T(t) sin\frac{2m\pi t}{T} dt \\ &= \int_T \left[A_0 + \sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T} + B_k sin\frac{2k\pi t}{T}\right)\right] sin\frac{2m\pi t}{T} dt \\ &= \sum_{k=1}^{\infty} A_k \int_T cos\frac{2k\pi t}{T} sin\frac{2m\pi t}{T} dt + \sum_{k=1}^{\infty} B_k \int_T sin\frac{2k\pi t}{T} sin\frac{2m\pi t}{T} dt\end{aligned} \qquad (18)$$

where $m\geq 1$, $m\in N$ and $\int_T A_0 \, sin\frac{2m\pi t}{T} dt = 0$. It is known that

$$\int_T cos\frac{2k\pi t}{T} sin\frac{2m\pi t}{T} dt = \frac{1}{2}\int_T \left(sin\frac{2(k+m)\pi t}{T} - sin\frac{2(k-m)\pi t}{T}\right) dt \qquad (19)$$

$$\int_T sin\frac{2k\pi t}{T} sin\frac{2m\pi t}{T} dt = \frac{1}{2}\int_T \left(cos\frac{2(k-m)\pi t}{T} - cos\frac{2(k+m)\pi t}{T}\right) dt \qquad (20)$$

where $\int_T cos\frac{2(k+m)\pi t}{T} dt = 0$, $\int_T sin\frac{2(k+m)\pi t}{T} dt = 0$, and $\int_T sin\frac{2(k-m)\pi t}{T} dt = 0$ for k, $m\in N$. Hence, (19) and (20) can be written into

$$\int_T cos\frac{2k\pi t}{T} sin\frac{2m\pi t}{T} dt = 0 \qquad (21)$$

$$\int_T sin\frac{2k\pi t}{T}sin\frac{2m\pi t}{T}dt=\frac{1}{2}\int_T cos\frac{2(k-m)\pi t}{T}dt=\begin{cases}T/2, & k=m\\ 0, & k\neq m\end{cases} \tag{22}$$

Therefore, the two summations in (18) can be evaluated as

$$\sum_{k=1}^{\infty}A_k\int_T cos\frac{2k\pi t}{T}sin\frac{2m\pi t}{T}dt=0 \tag{23}$$

$$\sum_{k=1}^{\infty}B_k\int_T sin\frac{2k\pi t}{T}sin\frac{2m\pi t}{T}dt=B_mT/2 \tag{24}$$

The truth of (23) is a result of (21), whose integration is zero for all $k\in N$. As for (24), it can be easily derived from (22), whose integration does not vanish only for $k=m$. Hence, from (18) we have

$$\int_T f_T(t)sin\frac{2m\pi t}{T}dt=B_mT/2 \qquad \text{for } m\in N \tag{25}$$

By changing m into k,we obtain

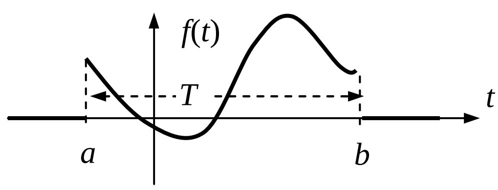
$$\int_T f_T(t)sin\frac{2k\pi t}{T}dt=B_kT/2 \qquad \text{for } k\in N \tag{26}$$

Then, the coefficient B_k in (4) is evaluated as

$$B_k=\frac{2}{T}\int_T f_T(t)sin\frac{2k\pi t}{T}dt \qquad \text{for } k\in N \tag{27}$$

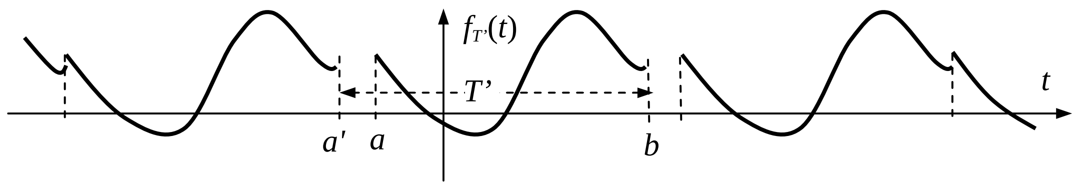
It is important to notice that any periodic function $f_T(t)$ satisfying the Dirichlet conditions C1~C3 can be also represented by its Fourier series coefficents A_0, A_k and B_k in (2), (3) and (4), which will extend to the concept of frequency spectrum.

Actually, Fourier series (1) can be also used to represent a function with finite time duration, such as the function $f(t)$ in the figure on the right. Since it is one part of $f_T(t)$ for $t\in(a,b)$, we have



$$f(t)=A_0+\sum_{k=1}^{\infty}\left(A_k cos\frac{2k\pi t}{T}+B_k sin\frac{2k\pi t}{T}\right) \qquad \text{for } t\in(a,b) \tag{28}$$

which is the same as (1) except the time duration is limited to $t\in(a, b)$. However, (28) is not a unique expression for the finite time duration function $f(t)$ due to the fact that it can be also treated as one part of the periodic function $f_{T'}(t)$ shown below:



With the period $T'=b-a'$, $f_{T'}(t)$ is represented by the Fourier series as

$$f_{T'}(t) = A'_0 + \sum_{k=1}^{\infty}\left(A'_k cos\frac{2k\pi t}{T'} + B'_k sin\frac{2k\pi t}{T'} \right) \qquad \text{for } t\in(-\infty,\infty) \quad (29)$$

which leads to

$$f(t) = A'_0 + \sum_{k=1}^{\infty}\left(A'_k cos\frac{2k\pi t}{T'} + B'_k sin\frac{2k\pi t}{T'} \right) \qquad \text{for } t\in(a,b) \quad (30)$$

From (28) and (30), we can conclude that the expression of a finite duration function in Fourier series is not unique. Later, we will apply the Fourier transform to deal with the problem.

Exercise:

Find the Fourier series of the following periodic functions:

(A) $f(t+4) = f(t)$ and $f(t) = f_T(t)$ for $0 < t < 4$

where $f_T(t) = \begin{cases} t-1, & 0<t<2 \\ -t+3, & 2<t<4 \end{cases}$

(B) $g(t+3\pi) = g(t)$ and $g(t) = g_T(t)$ for $0 < t < 2\pi$

where $g_T(t) = \left| sin\frac{2}{\pi}t \right|$

[補充資料] 傳立葉小傳

勁·巴帝斯·約瑟夫·傅立葉

Jean Baptiste Joseph Fourier

1768 年 3 月 21 日-1830 年 5 月 16 日

法國數學家、物理學家，研究熱傳導理論，以傅立葉級數聞名於世，《熱的分析理論》是其不朽的論文，傅立葉轉換因基本觀念來自傅立葉，故以其名命名以示紀念

傅立葉是裁縫的兒子，8 歲父母雙亡後，進入聖本篤修道院所辦的軍事學校，12 歲能幫神父寫佈道書，13 歲即顯現出數學才華，19 歲時成為見習神父，1789 年法國大革命給了他自由，被學校聘為數學教授，教學出色。傅立葉是革命的熱情參與者，26 歲曾因此入獄，隨後因政治氣氛改變而出獄。

1795 年傅立葉成為法國高等師範學院的第一批學生，教授群都是一時之選，包括拉格朗日(Lagrange)、拉普拉斯(Laplace)等。1798 他隨拿破崙征服埃及，之後回到巴黎，被拿破崙派任到格勒諾布爾(Grenoble)大學當校長，一展其行政長才，他一生中最重要的熱傳導理論也是在這裡完成的，他所提出的傅立葉級數，將函數轉化成三角函數級數和的方法，對日後的科技發展有重大的影響。

傅立葉的論文《固體中的熱傳導》曾獲得高度重視，但也因為缺乏嚴謹的證明，引起不少爭議與學術論戰，這個情況一直到 Dirichlet 登場，才真正落幕。