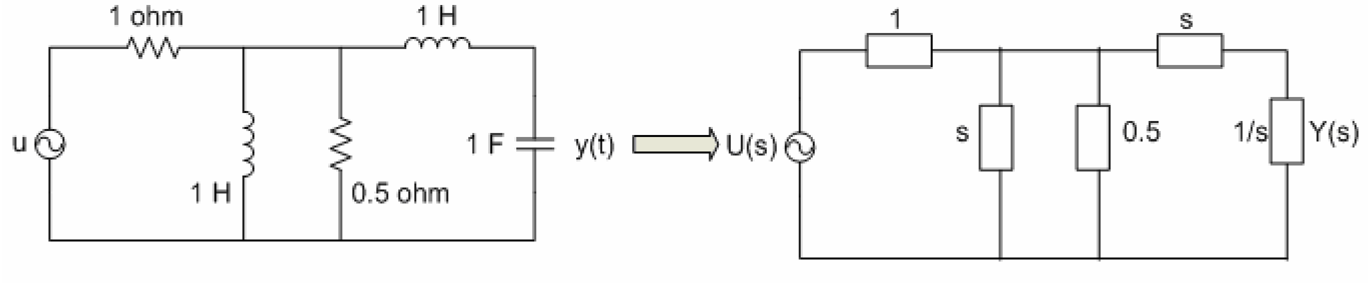
Signals And Systems Exam#2

1. (8%)



From the circuit above, we can find the relation between input and output in s-domain:

$$Y(s)=\frac{s//0.5//(s+1/s)}{1+[s//0.5//(s+1/s)]}\cdot\frac{1/s}{(s+1/s)}\cdot U(s)$$

$$(s//0.5)=\frac{0.5s}{s+0.5}\ ,\ (s+\frac{1}{s})=\frac{s^2+1}{s}\ ,\ \frac{1/s}{(s+1/s)}=\frac{1}{s^2+1}$$

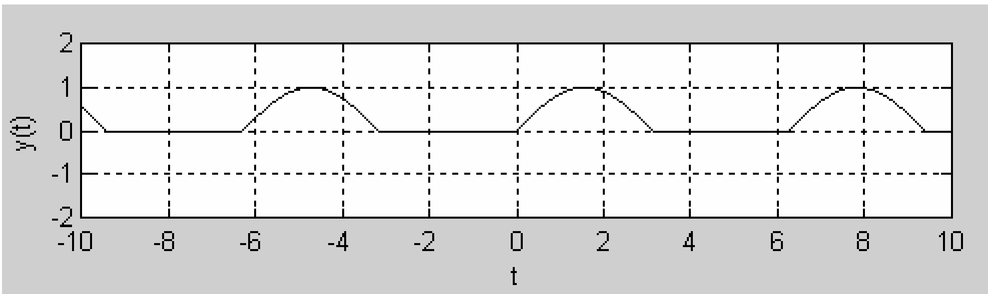
$$(s//0.5)//(s+\frac{1}{s})=\frac{0.5s}{s+0.5}//\frac{s^2+1}{s}=\frac{0.5s^3+0.5s}{s^3+s^2+s+0.5}$$

$$H(s)=\frac{\dfrac{0.5s^3+0.5s}{s^3+s^2+s+0.5}}{1+(\dfrac{0.5s^3+0.5s}{s^3+s^2+s+0.5})}\cdot\frac{1}{s^2+1}=\frac{0.5s(s^2+1)}{1.5s^3+s^2+1.5s+0.5}\cdot\frac{1}{s^2+1}=\frac{0.5s}{1.5s^3+s^2+1.5s+0.5}$$

2. (18%)

(A) Fundamental period $P=2\pi$

Fundamental frequency $\omega_o=\frac{2\pi}{P}=1$



(B) $P_{av}=\frac{1}{P}\int_0^P|y(t)|^2\,dt=\frac{1}{2\pi}\int_0^\pi\sin^2 t\,dt=\frac{1}{2\pi}\int_0^\pi\frac{1-\cos 2t}{2}dt=\frac{1}{2\pi}[\frac{1}{2}t-\frac{1}{4}\sin 2t]|_0^\pi=\frac{1}{4}$

(C) $y(t)=\sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t}$

$$c_m=\frac{1}{2\pi}\int_0^{\pi}\sin t\cdot e^{-jm\omega_o t}dt=\frac{1}{2\pi}\int_0^{\pi}\frac{e^{jt}-e^{-jt}}{2j}\cdot e^{-jm\omega_o t}dt \quad (\omega_o=1)$$

$$=\frac{1}{4\pi j}\int_0^{\pi}(e^{jt}-e^{-jt})\cdot e^{-jmt}dt=\frac{1}{4\pi j}\int_0^{\pi}(e^{-j(m-1)t}-e^{-j(m+1)t})dt$$

$$=\frac{1}{4\pi j}[\frac{e^{-j(m-1)t}}{-j(m-1)}-\frac{e^{-j(m+1)t}}{-j(m+1)}]|_0^{\pi}$$

$$=\frac{1}{4\pi j}[(\frac{e^{-j(m-1)\pi}}{-j(m-1)}-\frac{e^{-j(m+1)\pi}}{-j(m+1)})-(\frac{1}{-j(m-1)}-\frac{1}{-j(m+1)})]$$

$$=\begin{cases}0 & \text{,if m is odd}\\ \dfrac{1}{\pi(1-m^2)} & \text{,if m is even}\end{cases}$$

(D) $Y(\omega)=\sum_{m=-\infty}^{\infty} 2\pi c_m\delta(\omega-m\omega_o)=\sum_{m=-\infty}^{\infty} 2\pi c_m\delta(\omega-m)$

Where $c_m=\begin{cases}0 & \text{,if m is odd}\\ \dfrac{1}{\pi(1-m^2)} & \text{,if m is even}\end{cases}$

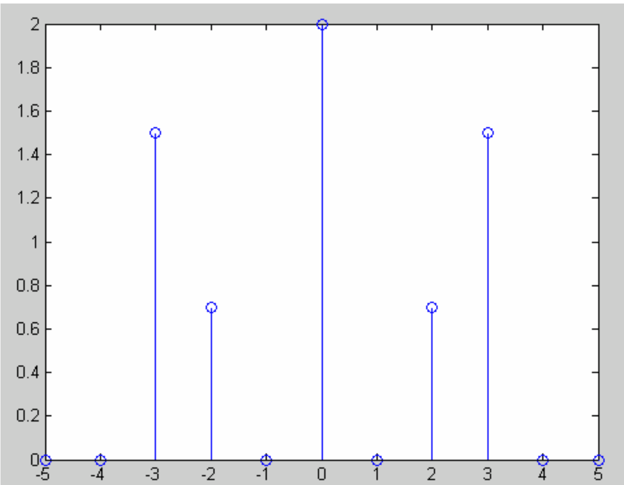
3. (10%)

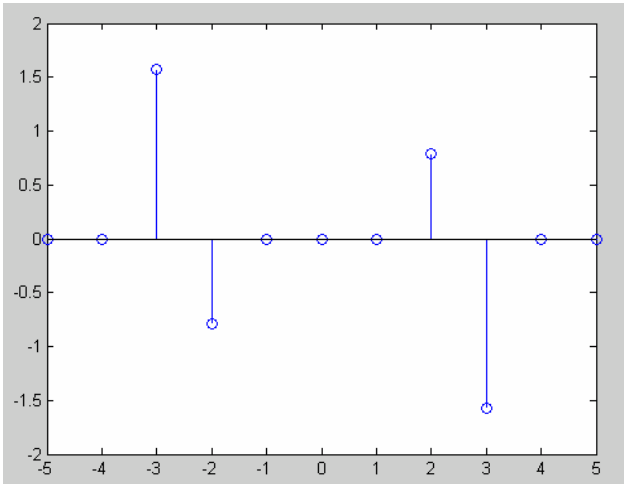
(A) $x(t)=2-\sin 2t+3\sin 3t+\cos 2t$

$$=2e^{j0t}-\frac{1}{2j}(e^{j2t}-e^{-j2t})+\frac{3}{2j}(e^{j3t}-e^{-j3t})+\frac{1}{2}(e^{j2t}+e^{-j2t})$$

$$=2e^{j0t}+(\frac{1+j}{2})\cdot e^{j2t}+(\frac{1-j}{2})\cdot e^{-j2t}+(-\frac{3}{2}j)\cdot e^{j3t}+(\frac{3}{2}j)\cdot e^{-j3t}$$

$$=2e^{j0}e^{j0t}+\frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}e^{j2t}+\frac{\sqrt{2}}{2}e^{-j\frac{\pi}{4}}e^{-j2t}+\frac{3}{2}e^{-j\frac{\pi}{2}}e^{j3t}+\frac{3}{2}e^{j\frac{\pi}{2}}e^{-j3t}$$



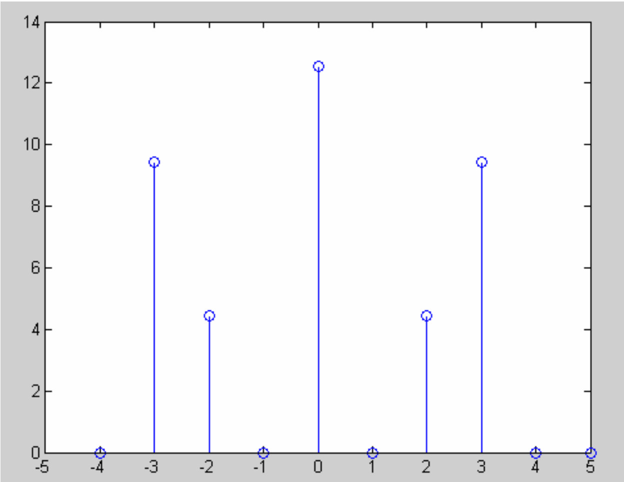


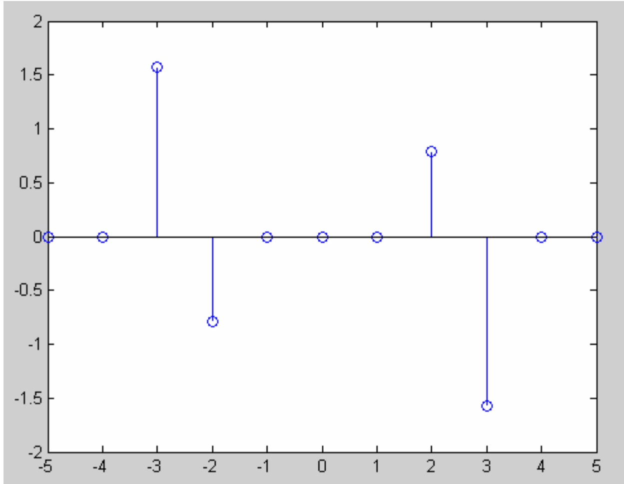
or

$$x(t) = 2 - \sin 2t + 3\sin 3t + \cos 2t$$

$$\Rightarrow X(\omega) = 2\cdot 2\pi\delta(\omega) - \frac{2\pi\delta(\omega-2) - 2\pi\delta(\omega+2)}{2j} + 3\cdot\frac{2\pi\delta(\omega-3) - 2\pi\delta(\omega+3)}{2j} + \frac{2\pi\delta(\omega-2) + 2\pi\delta(\omega+2)}{2}$$

$$= 4\pi\delta(\omega) + \sqrt{2}\pi e^{j\frac{\pi}{4}}\delta(\omega-2) + \sqrt{2}\pi e^{-j\frac{\pi}{4}}\delta(\omega+2) + 3\pi e^{-j\frac{\pi}{2}}\delta(\omega-3) + 3\pi e^{j\frac{\pi}{2}}\delta(\omega+3)$$

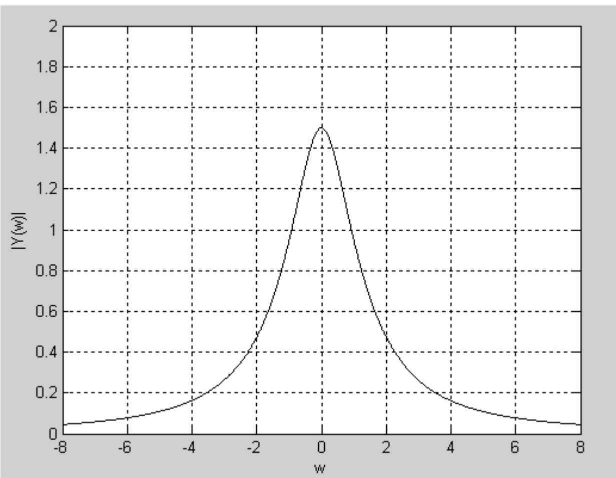


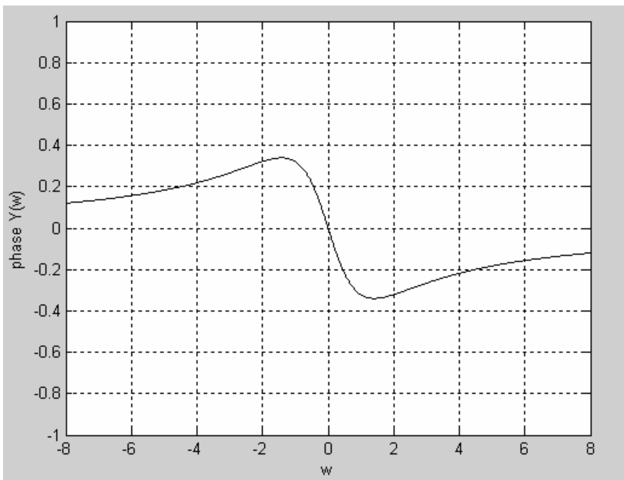


(B) $Y(\omega) = \int_{-\infty}^{\infty} y(t)\cdot e^{-j\omega t}dt = \int_{-\infty}^{0} e^{2t}\cdot e^{-j\omega t}dt + \int_{-\infty}^{0} e^{-t}\cdot e^{-j\omega t}dt$

$$= \frac{e^{(2-j\omega)t}}{2-j\omega}\Big|_{-\infty}^{0} + \frac{e^{(-1-j\omega)t}}{-1-j\omega}\Big|_{0}^{\infty} = \frac{1}{2-j\omega} + \frac{1}{1+j\omega} = \frac{3}{(\omega^2+2)+j\omega}$$

$$|Y(\omega)| = \frac{3}{\sqrt{(\omega^2+2)^2+\omega^2}} \quad , \quad \angle Y(\omega) = -\tan^{-1}\frac{\omega}{\omega^2+2}$$





4. (12%)

(A) $F[g(t)\cos 20t] = \frac{1}{2\pi}G(\omega) * (\pi\delta(\omega-20) + \pi\delta(\omega+20))$

$$= \frac{1}{2}[\mathrm{G}(\omega-20) + G(\omega+20)]$$

(B) $F[(g(t)\cdot\cos^2 200t) * h(t)] = F[g(t)\cdot\cos^2 200t]\cdot H(\omega)$

$$= F[g(t)\cdot\frac{1+\cos 400t}{2}]\cdot H(\omega) = F[\frac{g(t)}{2}]\cdot H(\omega) + F[\frac{g(t)\cdot\cos 400t}{2}]\cdot H(\omega)$$

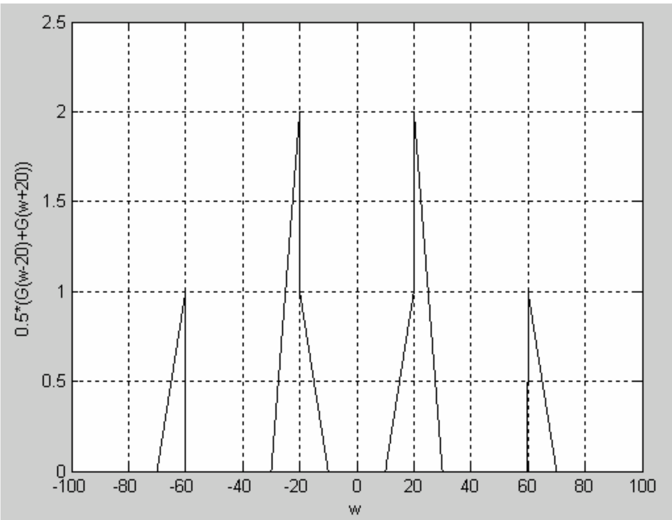
$$= \frac{1}{2}G(\omega)H(\omega) \quad (\because F[\frac{g(t)\cdot\cos 400t}{2}]\cdot H(\omega) = 0)$$

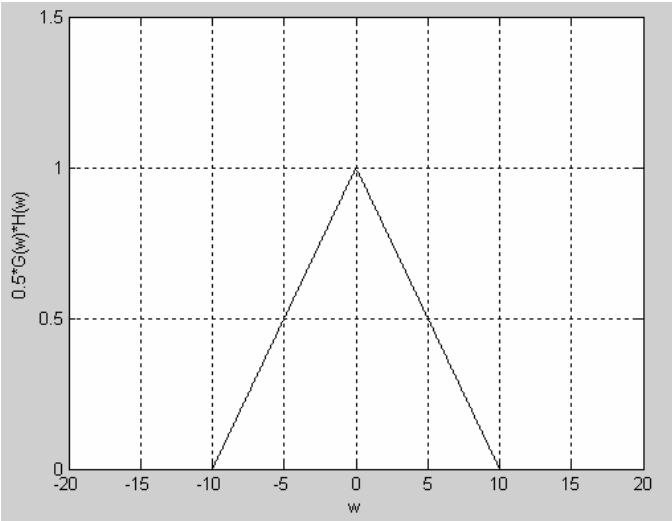
Total energy: $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega$

$$E = \frac{1}{2\pi}[\int_{-10}^{0} (\frac{1}{10}\omega+1)^2 d\omega + \int_{0}^{10} (-\frac{1}{10}\omega+1)^2 d\omega]$$

$$= \frac{1}{2\pi}[(\frac{1}{300}\omega^3 + \frac{1}{10}\omega^2 + \omega)|_{-10}^{0} + (\frac{1}{300}\omega^3 - \frac{1}{10}\omega^2 + \omega)|_{0}^{10}]$$

$$= \frac{1}{2\pi}[\frac{10}{3} + \frac{10}{3}] = \frac{10}{3\pi}$$





5. (14%)

(A) $H(s) = \frac{Y(s)}{U(s)} = \frac{2s^3+4s^2+3s}{s^4+4s^3+7s^2+6s+2}$

$$\Rightarrow Y(s)(s^4+4s^3+7s^2+6s+2) = U(s)(2s^3+4s^2+3s)$$

$$\Rightarrow y^{(4)}(t)+4y^{(3)}(t)+7y^{(2)}(t)+6y'(t)+2y(t) = 2u^{(3)}(t)+4u^{(2)}(t)+3u'(t)$$

(B) $H(s) = \frac{2s^3+4s^2+3s}{s^4+4s^3+7s^2+6s+2} = \frac{2s^3+4s^2+3s}{(s+1)^2(s^2+2s+2)}$

$$= \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k_3(s+1)}{[(s+1)^2+1]} + \frac{k_4}{[(s+1)^2+1]}$$

$$k_2 = Y(s)(s+1)^2 |_{s=-1} = -1$$

$$s=0 \Rightarrow Y(s)=0=k_1-1+\frac{k_3}{2}+\frac{k_4}{2}$$

$$s=1 \Rightarrow Y(s)=\frac{9}{20}=\frac{k_1}{2}-\frac{1}{4}+\frac{2k_3}{5}+\frac{k_4}{5} \quad \Rightarrow \begin{cases} 2k_1+k_3+k_4=2 \\ 5k_1+4k_3+2k_4=7 \\ 2k_1+k_3-k_4=4 \end{cases} \Rightarrow k_1=1,\ k_3=1,\ k_4=-1$$

$$s=-2 \Rightarrow Y(s)=-3=-k_1-1+\frac{-k_3}{2}+\frac{k_4}{2}$$

$$\Rightarrow H(s)=\frac{1}{s+1}+\frac{-1}{(s+1)^2}+\frac{(s+1)}{[(s+1)^2+1]}+\frac{-1}{[(s+1)^2+1]}$$

$$\Rightarrow h(t)=e^{-t}-te^{-t}+e^{-t}\cos t-e^{-t}\sin t \quad , for\ t\ge 0$$

6. (20%)

(A) $U(s)=\frac{1}{s^2+1} \qquad Y(s)=\frac{1}{2}\cdot\frac{1}{s+1}-\frac{1}{s+2}+\frac{1}{2}\cdot\frac{1}{s+3}=\frac{1}{(s+1)(s+2)(s+3)}$

$$\Rightarrow H(s)=\frac{Y(s)}{U(s)}=\frac{s^2+1}{(s+1)(s+2)(s+3)}=\frac{1}{s+1}+\frac{-5}{s+2}+\frac{5}{s+3}$$

$$\Rightarrow h(t)=e^{-t}-5e^{-2t}+5e^{-3t}$$

(B) $H(s)=\frac{Y(s)}{U(s)}=\frac{s^2+1}{s^3+6s^2+11s+6}$

$$\Rightarrow [s^3Y(s)-s^2y(0)-sy'(0)-y^{(2)}(0)]+6[s^2Y(s)-sy(0)-y'(0)]+11[sY(s)-y(0)]+6Y(s)$$
$$=[s^2U(s)-su(0)-u'(0)]+U(s)$$

$$\Rightarrow [s^3Y(s)-s^2]+6[s^2Y(s)-s]+11[sY(s)-1]+6Y(s)=[s^2U(s)-s]+U(s)$$

$$\Rightarrow Y(s)(s^3+6s^2+11s+6)=U(s)(s^2+1)-s+(s^2+6s+11)$$

$$\Rightarrow Y(s)(s^3+6s^2+11s+6)=(\frac{s}{s^2+1})(s^2+1)+(s^2+5s+11)=s^2+6s+11$$

$$\Rightarrow Y(s)=\frac{s^2+6s+11}{s^3+6s^2+11s+6}=\frac{3}{s+1}+\frac{-3}{s+2}+\frac{1}{s+3}$$

$$\Rightarrow y(t)=3e^{-t}-3e^{-2t}+e^{-3t} \quad , for\ \ t\ge 0$$

(C) From the equation above:

$$[s^3Y(s)-s^2y(0)-sy'(0)-y^{(2)}(0)]+6[s^2Y(s)-sy(0)-y'(0)]+11[sY(s)-y(0)]+6Y(s)$$
$$=[s^2U(s)-su(0)-u'(0)]+U(s)$$
$$\Rightarrow Y(s)(s^3+6s^2+11s+6)=U(s)(s^2+1)-s+y(0)\cdot(s^2+6s+11)+y'(0)\cdot(s+6)+y^{(2)}(0)$$

Let $U(s)=\frac{s}{s^2+1} \qquad Y(s)=\frac{1}{s^3+6s^2+11s+6}$

$$\Rightarrow Y(s)(s^3+6s^2+11s+6)=s-s+y(0)\cdot(s^2+6s+11)+y'(0)\cdot(s+6)+y^{(2)}(0)=1$$

$$\Rightarrow y(0)=0\ ,\ y'(0)=0\ ,\ y^{(2)}(0)=1$$

Yes! We can find the initial conditions

7. (18%)

(A) $f_1(t)=\frac{1}{2\pi}\int_{-3}^{3}2e^{j\omega t}d\omega=\frac{1}{\pi}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-3}^{3}=\frac{2}{\pi t}\cdot[\frac{1}{2j}(e^{j3t}-e^{-j3t})]=\frac{2\sin 3t}{\pi t}$

(B) $\because F[x^{'}(t)]=j\omega X(\omega)$

$\therefore f_2(t)=f_1^{'}(t)=\frac{6t\cos 3t-2\sin 3t}{\pi t^2}$

(C) $\because F[e^{j\omega_o t}x(t)]=X(\omega-\omega_o)$

$\therefore f_3(t)=e^{jt}f_1(t)=e^{jt}\frac{2\sin 3t}{\pi t}$

(D) $\because F[x(t+t_o)]=e^{j\omega t_o}X(\omega)$

$\therefore f_4(t)=f_1(t+1)=\frac{2\sin 3(t+1)}{\pi(t+1)}$