Signals And Systems Final Exam

1. (25%)

(A) In order to find the minimum phase system, we have to check both poles and zeroes are located in the left-half complex plane. Routh's test can do great help for checking them.

(i) Test for $H_A(s) = \dfrac{s^2+6s+5}{s^3+s^2+4s+9}$

Check for stability:

s^3	1	4
s^2	1	9
s^1	-5 → negative	

$\Rightarrow$ Because it is an unstable system, it can't be minimum phase system

(ii) Test for $H_B(s) = \dfrac{s^3+s^2+4s+9}{s^4+9s^3+26s^2+34s+20}$

Check for stability:

s^4	1	26	20
s^3	9	34	
s^2	$\frac{200}{9}$	20	
s^1	25.9		
s^0	20		

$\Rightarrow$ Stable

Check for minimum phase:

s^3	1	4
s^2	1	9
s^1	-5 → negative	

$\Rightarrow$ Because it doesn't pass the Routh's test, there are some zeroes at RHP and is non-minimum phase system

(iii) Test for $H_C(s) = \dfrac{s^3+4s^2+4s+8}{s^4+9s^3+26s^2+34s+20}$

Stability: Stable (characteristic function is the same as above)

Check for minimum phase:

s^3	1	4
s^2	4	8
s^1	2	
s^0	8	

$\Rightarrow$ all zeroes are in the LHP

$\Rightarrow$ Because all the poles and zeros are in the LHP, **H_C is minimum phase system**

(B) $H_3 = \dfrac{H_A}{1+kH_A} = \dfrac{\dfrac{s^2+6s+5}{s^3+s^2+4s+9}}{1+k\cdot\dfrac{s^2+6s+5}{s^3+s^2+4s+9}} = \dfrac{s^2+6s+5}{s^3+(1+k)s^2+(4+6k)s+(9+5k)}$

From Routh's test

s^3	1	$4+6k$
s^2	$1+k$	$9+5k$
s^1	$(4+6k)-\dfrac{9+5k}{1+k}$	
s^0	$9+5k$	

In order to make the system stable, we must let all the coefficients in Routh table > 0

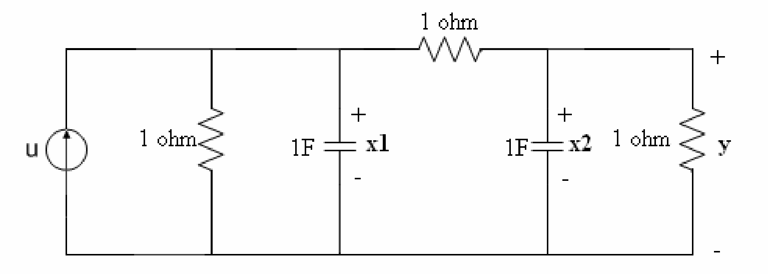
$$\Rightarrow \begin{cases} 4+6k>0 \\ 1+k>0 \\ 9+5k>0 \\ \dfrac{6k^2+5k-5}{1+k}>0 \end{cases} \Rightarrow \begin{cases} k>-\dfrac{2}{3} \\ k>-1 \\ k>-\dfrac{9}{5} \\ k>\dfrac{-5+\sqrt{145}}{12}\text{(o)} \quad or \quad k<\dfrac{-5-\sqrt{145}}{12}\text{(x)} \end{cases} \Rightarrow k>\frac{-5+\sqrt{145}}{12}\simeq 0.587$$

(C) With $k=1$, $H_3(s)=\dfrac{s^2+6s+5}{s^3+(1+k)s^2+(4+6k)s+(9+5k)}\Big|_{s=1}=\dfrac{s^2+6s+5}{s^3+2s^2+10s+14}$

Canonical form:

$$\underline{\dot{x}} = \begin{bmatrix} -2 & -10 & -14 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u \quad , \quad y=[1 \quad 6 \quad 5]\underline{x}$$

2. (28%)



(A)

By KCL： $u=\dfrac{x_1}{1}+1\cdot\dfrac{dx_1}{dt}+\dfrac{x_1-x_2}{1}=x_1+\dot{x}_1+x_1-x_2=\dot{x}_1+2x_1-x_2$

$\Rightarrow \dot{x}_1=-2x_1+x_2+u$ ……(1)

By KVL： $x_1=x_2+1\cdot(\dfrac{x_2}{1}+\dfrac{dx_2}{dt})=x_2+x_2+\dot{x}_2=2x_2+\dot{x}_2$

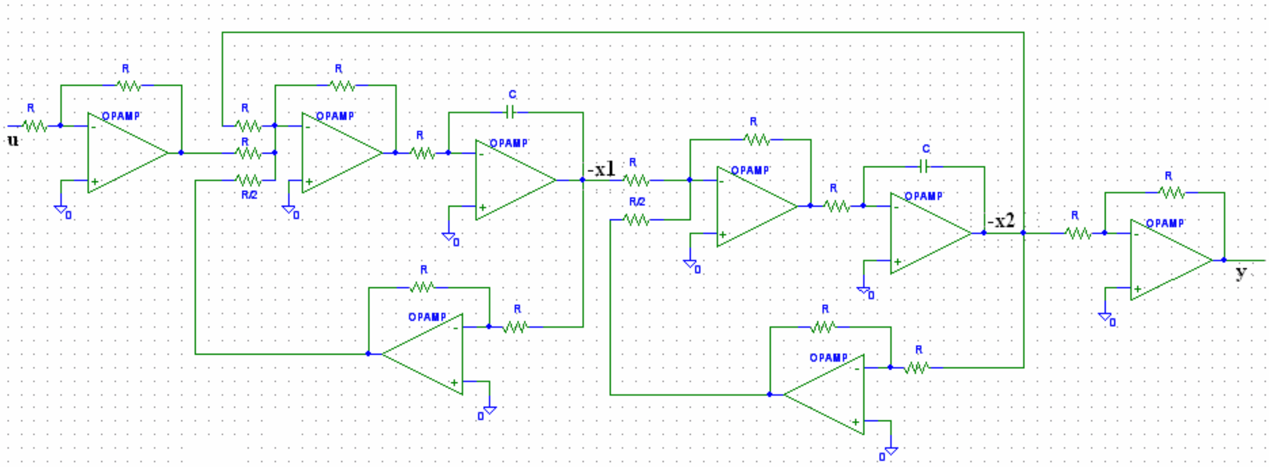
$\Rightarrow \dot{x}_2=x_1-2x_2$ ……(2)

And $y=x_2$ ……(3)

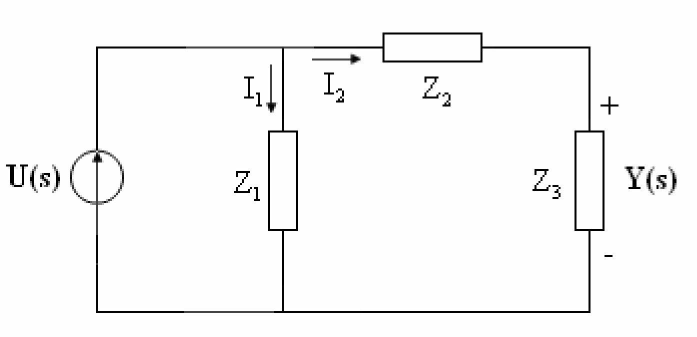
By (1)、(2)、(3) we can construct state-space description matrices：

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

(B)



(C)



$$Z_1 = Z_3 = 1 // \frac{1}{s} = \frac{1}{s+1} \quad , \quad Z_2 = 1$$

$$I_2(s) = \frac{Z_1}{Z_1 + (Z_2 + Z_3)} \cdot U(s) = \frac{\frac{1}{s+1}}{(\frac{1}{s+1}) + (1 + \frac{1}{s+1})} \cdot U(s) = \frac{\frac{1}{s+1}}{\frac{1}{s+1} + \frac{s+2}{s+1}} \cdot U(s) = \frac{1}{s+3} U(s)$$

$$Y(s) = I_2(s) \cdot Z_3 = [\frac{1}{s+3} U(s)] \cdot \frac{1}{s+1} = \frac{1}{s^2 + 4s + 3} U(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 4s + 3}$$

3.(18%)

(A)

$$u(t)=A\cos\omega t\Rightarrow U(s)=\frac{As}{s^2+\omega^2}$$

$$Y(s)=H(s)\cdot U(s)=H(s)\cdot\frac{As}{s^2+\omega^2}=\frac{k_1}{s+j\omega}+\frac{k_2}{s-j\omega}+\ \textit{terms due to poles of } H(s)$$

$$\Rightarrow\begin{cases}k_1=\dfrac{A\cdot H(-j\omega)}{2}\\ k_2=\dfrac{A\cdot H(j\omega)}{2}\end{cases}$$

$$\Rightarrow Y(s)=\frac{1}{s+j\omega}\cdot\frac{AH(-j\omega)}{2}+\frac{1}{s-j\omega}\cdot\frac{AH(j\omega)}{2}+\ \textit{terms due to poles of } H(s)$$

$$=\frac{1}{s+j\omega}\cdot\frac{A|H(j\omega)|e^{-j\angle H(j\omega)}}{2}+\frac{1}{s-j\omega}\cdot\frac{A|H(j\omega)|e^{j\angle H(j\omega)}}{2}+\ \textit{terms due to poles of } H(s)$$

Because the terms due to poles of $H(s)$ will decay to zero when $t\to\infty$

$$\Rightarrow y_{ss}(t)=A|H(j\omega)|\cdot(\frac{e^{-j\omega t-j\angle H(j\omega)}+e^{j\omega t+j\angle H(j\omega)}}{2})=A|H(j\omega)|\cdot(\frac{e^{-j(\omega t+\angle H(j\omega))}+e^{j(\omega t+\angle H(j\omega))}}{2})$$

$$\Rightarrow y_{ss}(t)=A|H(j\omega)|\cos(\omega t+\angle H(j\omega))$$

(B)

$$u=2\cos 2t-\sin(t-1)$$

$$H_3(s)=\frac{s^2+6s+5}{s^3+2s^2+4s+9}\Rightarrow H_3(j\omega)=\frac{-\omega^2+j6\omega+5}{-j\omega^3-2\omega^2+j10\omega+14}=\frac{(-\omega^2+5)+j6\omega}{(-2\omega^2+14)+j(-\omega^3+10\omega)}$$

(i) *when* $\omega=1$:

$$H_3(j1)=\frac{4+6j}{12+9j}\Rightarrow\ |H(j1)|\cong 0.47\ ,\ \angle|H(j1)|\cong 0.34$$

(ii) *when* $\omega=2$:

$$H_3(j2)=\frac{1+12j}{6+12j}\Rightarrow\ |H(j2)|\cong 0.9\ ,\ \angle|H(j2)|\cong 0.38$$

From $(i),(ii)$ and the result of (A), we can find the steady-state output $y_{ss}(t)$:

$$y_{ss}(t)=2\cdot 0.9\cos(2t+0.38)-0.48\sin(t-1+0.34)$$
$$=1.8\cos(2t+0.38)-0.48\sin(t-0.66)$$

4.(29%)

(A) $H_L(j\omega)=\begin{cases}e^{-j0.4\omega}, & |\omega|\le 3\\ 0\quad, & |\omega|\ge 3\end{cases}$

(B) *input* $u(t)=\sin 1.2t+\cos 2.4t+\cos 3.6t$

output $y_{ss}(t)=1\cdot|H(j1.2)|\sin(1.2t+\angle H(j1.2))+1\cdot|H(j2.4)|\cos(2.4t+\angle H(j2.4))$

$+1\cdot|H(j3.6)|\sin(3.6t+\angle H(j3.6))$

$=1\cdot1\cdot\sin(1.2t-0.4\cdot1.2)+1\cdot1\cdot\cos(2.4t-0.4\cdot2.4)+1\cdot0$

$=\sin(1.2t-0.48)+\cos(2.4t-0.96)$

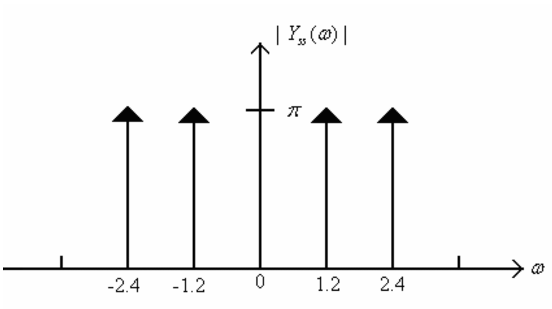
(C) $y_{ss}(t) = \sin(1.2t - 0.48) + \cos(2.4t - 0.96)$

$$\Rightarrow Y_{ss}(\omega) = e^{-j0.4\omega}[(-j\pi\delta(\omega-1.2) + j\pi\delta(\omega+1.2)) + (\pi\delta(\omega-2.4) + \pi\delta(\omega+2.4))]$$

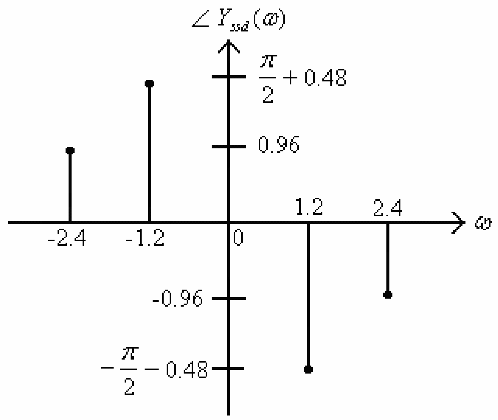
And, $Y_{ssd}(\omega) = \dfrac{1}{T}\sum_{n=-\infty}^{\infty} Y_{ss}(\omega - n\dfrac{2\pi}{T})$

(*i*) $Y_{ss}(\omega)$

Magnitude spectra:

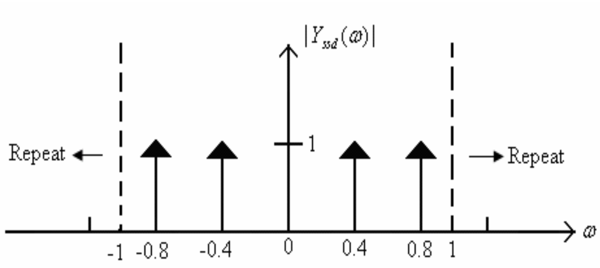


Phase spectra:

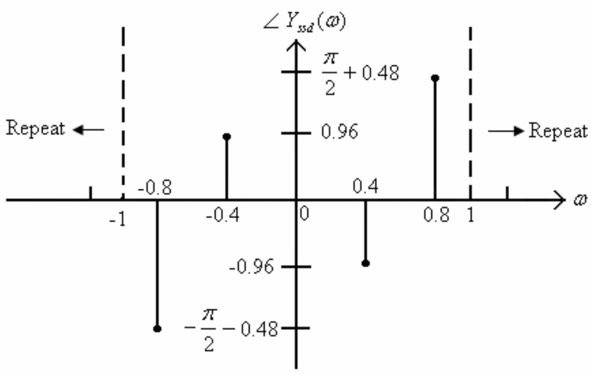


(*ii*) $Y_{ssd}(\omega)|_{T=\pi}$

Magnitude spectra:

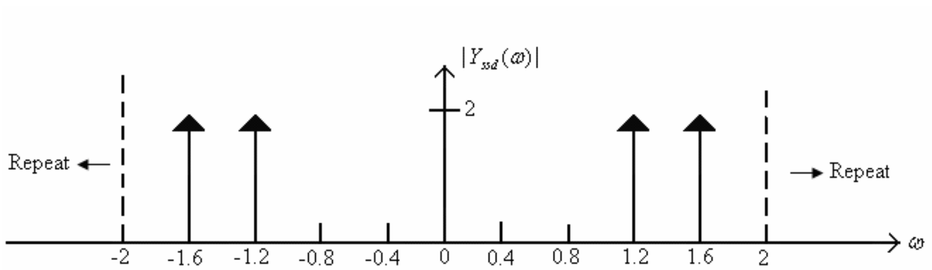


Phase spectra:

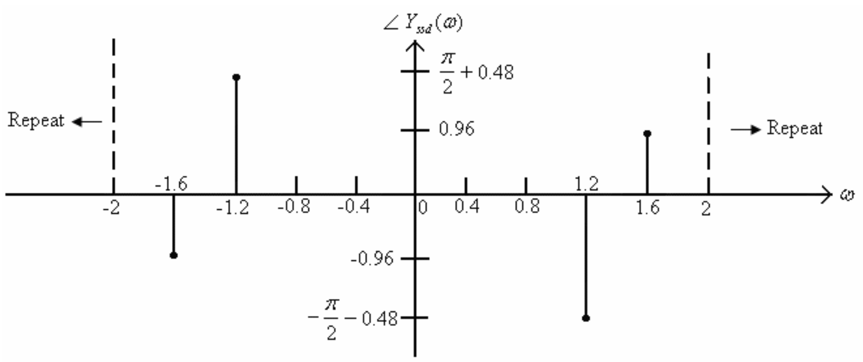


(*iii*) $Y_{ssd}(\omega)|_{T=\pi/2}$

Magnitude spectra:

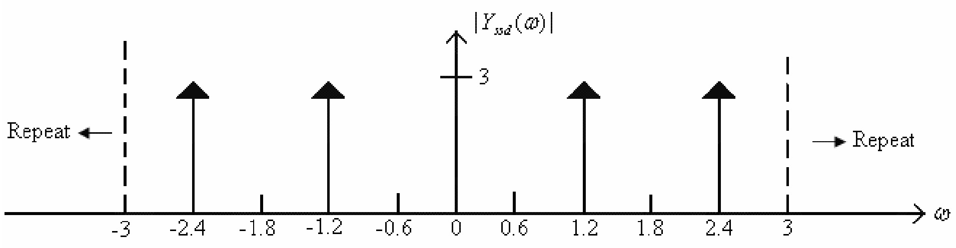


Phase spectra:

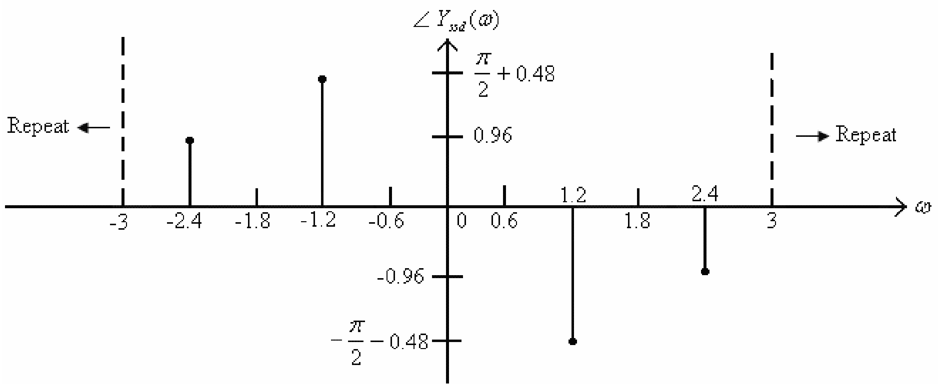


$(iv)\ Y_{ssd}(\omega)|_{T=\pi/3}$

Magnitude spectra:



Phase spectra:



(D) With the use of $H_L(s)$ as a low-pass filter, it will limit the frequency range within $[-3,3]$. From the spectra plotted above, we can see that when $T=\pi$ and $T=\frac{\pi}{2}$, the aliasing will happen. Only when $T=\frac{\pi}{3}$, $y_{ss}(t)$ can be recovered. Therefore, it is possible to recover $y_{ss}(t)$ from the discrete signals $y[n]$ obtained in (C) if we choose the sampling period as $T=\frac{\pi}{3}$