2005 Signals and Systems Midterm 2 Solution

1.(A) $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{4}\cos 3t + \frac{3}{4}\cos t\right)e^{-j2kt}dt\,,\quad \omega_0 = \frac{2\pi}{T} = 2\,,\quad T = \pi$$

$$= \frac{1}{8\pi}\int_{-\pi/2}^{\pi/2}\left[e^{j(3-2k)t} + e^{j(-3-2k)t} + 3e^{j(1-2k)t} + 3e^{j(-1-2k)t}\right]dt$$

$$= \frac{3(-1)^k}{2\pi}\left(\frac{1}{4k^2-9} + \frac{-1}{4k^2-1}\right),\quad e^{jk\pi} = e^{-jk\pi} = (-1)^k$$

(B) $a_k = \frac{1}{3}\int_1^3 e^{-jk\frac{2\pi}{3}t}dt\,,\quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$

$$= \frac{1}{-jk2\pi}\left(1 - e^{-jk\frac{2\pi}{3}}\right)$$

2.(A) $X(j\omega) = \int_{-\infty}^{0} e^{2t}\sin 2t e^{-j\omega t}dt$

$$= \frac{1}{2j}\int_{-\infty}^{0}\left[e^{(2+j(2-\omega))t} - e^{(2-j(2+\omega))t}\right]dt$$

$$= \frac{1}{2j}\left(\frac{1}{2-j(\omega-2)} - \frac{1}{2-j(\omega+2)}\right)$$

(B) $X(j\omega) = \int_{-2}^{0} e^{-j\omega t}dt - \frac{1}{2}\int_0^2 te^{-j\omega t}dt$

$$= \frac{1}{j\omega}\left(e^{j2\omega} - 1\right) + \frac{1}{j\omega}e^{-j2\omega} - \frac{1}{2\omega^2}e^{-j2\omega} + \frac{1}{2\omega^2}$$

3.(A) $X(j\omega) = j\sin 2\omega \sin c\omega$

$$= \frac{1}{2}e^{j2\omega}\sin c\omega - \frac{1}{2}e^{-j2\omega}\sin c\omega$$

$$F^{-1}\{\sin c\omega\} = \frac{1}{2}\left[u(t+1) - u(t-1)\right]$$

$$F^{-1}\left\{e^{-j\omega t_0}X(j\omega)\right\} = x(t-t_0)$$

$$x(t) = \frac{1}{4}\left[u(t+3) - u(t+1) - u(t-1) + u(t-3)\right]$$

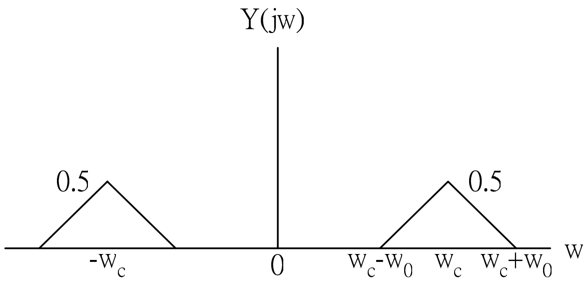
(B) $F^{-1}\{u(\omega+\omega_b) - u(\omega-\omega_b)\} = \frac{\omega_b}{\pi}\sin c\omega_b t$

$$x(t) = -\frac{3}{\pi}\sin c3t$$

4. $F\{\cos\omega_c t\} = \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$$

$$= \frac{1}{2}[X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))]$$



$$\omega_c - \omega_0 \geq -\omega_c + \omega_0$$

$$\omega_c \geq \omega_0$$

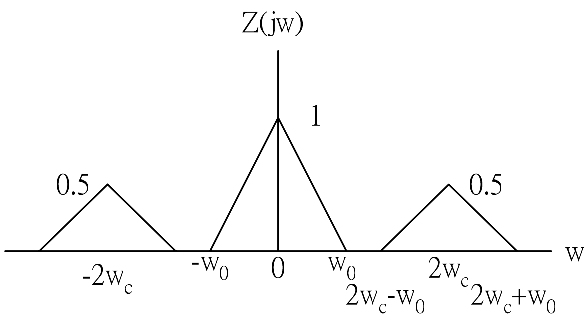
Demodulation technique:

$$z(t) = 2y(t)\cos\omega_c t$$

$$= 2x(t)\cos^2\omega_c t$$

$$= x(t) + \frac{1}{2}x(t)e^{j2\omega_c t} + \frac{1}{2}x(t)e^{-j2\omega_c t}$$

$$Z(j\omega) = X(j\omega) + \frac{1}{2}[X(j(\omega-2\omega_c)) + X(j(\omega+2\omega_c))]$$



5. $|x(t)|^2 = x(t)x^*(t)$

$$F\{x(t)x^*(t)\} = \frac{1}{2\pi} X(j\omega) * X^*(-j\omega)$$

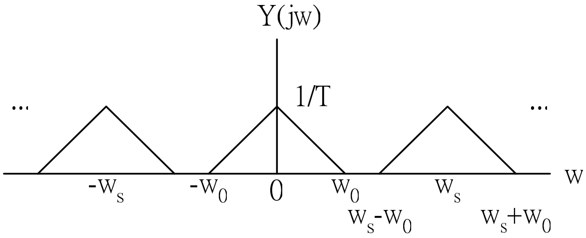
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$g(t) = |x(t)|^2$$

$$G(j\omega) = F\{g(t)\}$$

$$E_x = G(0)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\lambda)X^*(j\lambda)d\lambda$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\lambda)|^2 d\lambda$$
$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$$

6. $a_k = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk\omega_s t}dt\ ,\quad \omega_s = \frac{2\pi}{T}$
$$= \frac{1}{T}$$
$$P(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_s)$$
$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$
$$= \omega_s\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$
$$y(t) = x(t)p(t)$$
$$Y(j\omega) = \frac{1}{2\pi}X(j\omega)*P(j\omega)$$
$$= \frac{1}{2\pi}X(j\omega)*\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}X(j(\omega - k\omega_s))$$



$\omega_s - \omega_0 \geq \omega_0$

$\omega_s \geq 2\omega_0$ without aliasing

LPF which recover $x(t)$:

$$H_R(j\omega) = T[u(\omega + \omega_0) - u(\omega - \omega_0)]$$
$$h_R(t) = T\frac{\omega_0}{\pi}\sin c\omega_0 t$$

7. $X(j\omega) = \frac{1}{1 + j\omega}$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$= \frac{1}{4}\left[\frac{1}{(1+j\omega)^2} + \frac{-2}{(1+j\omega)(10+j\omega)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{(1+j\omega)^2} + \frac{-2/9}{1+j\omega} + \frac{2/9}{10+j\omega}\right]$$

$$y(t) = \frac{1}{4}\left(te^{-t} - \frac{2}{9}e^{-t} + \frac{2}{9}e^{-10t}\right)u(t)$$