2005 Signals and Systems Midterm 1 Solution

1.(1) $\frac{dx(t)}{dt} = \delta(t+1) - t\delta(t-1) - u(t-1) + t\delta(t-2) + u(t-2)$

$$= \delta(t+1) - \delta(t-1) - u(t-1) + 2\delta(t-2) + u(t-2)$$

$$\left(x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)\right)$$

(2) $\int_{-\infty}^{t} x(\tau)u(\tau)d\tau$

$$= \int_{-\infty}^{t} u(\tau)u(\tau+1)d\tau - \int_{-\infty}^{t} \tau u(\tau-1)u(\tau)d\tau + \int_{-\infty}^{t} \tau u(\tau-2)u(\tau)d\tau$$

$$= u(t)\int_{0}^{t} 1 d\tau - u(t-1)\int_{1}^{t} \tau d\tau + u(t-2)\int_{2}^{t} \tau d\tau$$

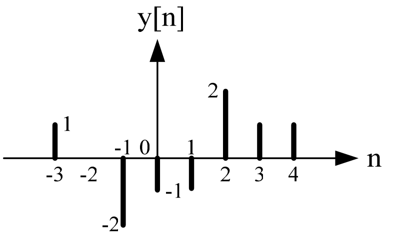
$$= tu(t) - \frac{1}{2}\left(t^2-1\right)u(t-1) + \frac{1}{2}\left(t^2-4\right)u(t-2)$$

(3) $Od\{x(t)\}$

$$= \frac{1}{2}\left[x(t) - x(-t)\right]$$

$$= \frac{1}{2}\left[u(t+1) - tu(t-1) + tu(t-2) - u(-t+1) - tu(-t-1) + tu(-t-2)\right]$$

2.



3. $t < -2$, $y(t) = 0$

$-2 < t < -1$, $y(t) = \int_{-1}^{t+1} (\tau+1)d\tau$

$$= \frac{1}{2}t^2 + 2t + 2$$

$-1 < t < 0$, $y(t) = -\int_{-1}^{t} (\tau+1)d\tau + \int_{t}^{t+1} (\tau+1)d\tau$

$$= -\frac{1}{2}t^2 + 1$$

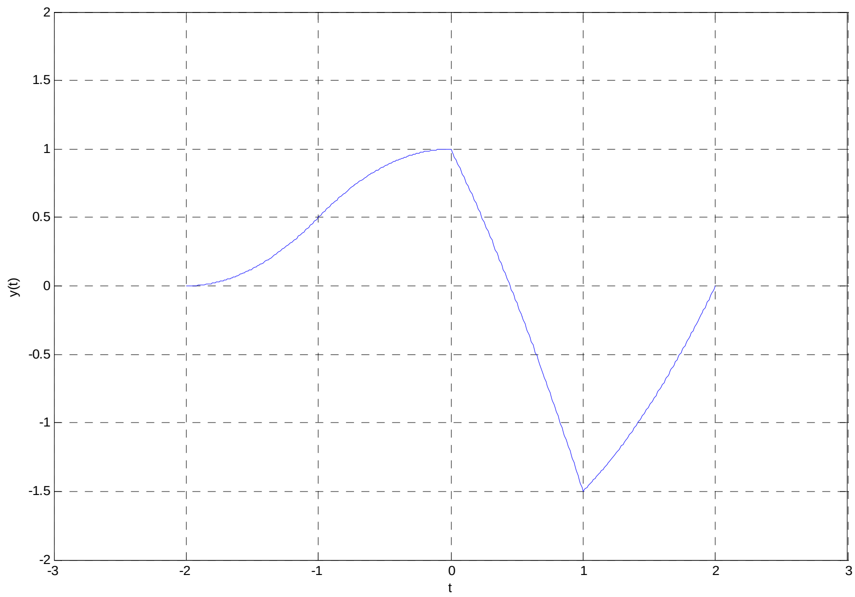
$0 < t < 1$, $y(t) = -\int_{t-1}^{t} (\tau+1)d\tau + \int_{t}^{1} (\tau+1)d\tau$

$$= -\frac{1}{2}t^2 - 2t + 1$$

$1 < t < 2$, $y(t) = -\int_{t-1}^{1}(\tau + 1)d\tau$

$$= \frac{1}{2}t^2 - 2$$

$2 < t$, $y(t) = 0$



4. Assume that $y_p(t) = A\cos t + B\sin t$, further we know $\dot{y}_p(t) = \text{-}A\sin t + B\cos t$ and $\ddot{y}_p(t) = -A\cos t - B\sin t$. After substituting the preceding assumption into the differential equation, we obtained

$$-A\cos t - B\sin t - 5A\sin t + 5B\cos t + 6A\cos t + 6B\sin t = 15\cos t - 5\sin t.$$

$A = 2$ and $B = 1$ could be known via the coefficients comparison and $y_p(t)$ was also obtained:

$$y_p(t) = 2\cos t + \sin t.$$

Furthermore, set $y_h(t) = Ke^{st}$ and substitute it into the corresponding homogeneous equation, we could get the characteristic equation of s:

$$s^2 + 5s + 6 = 0.$$

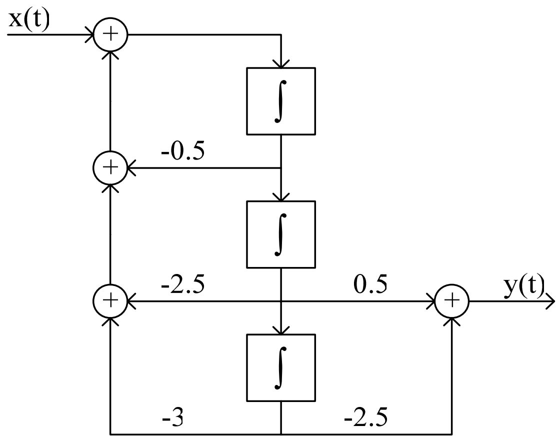
Then the characteristic equation was solved and known that $s = -2, -3$, so the solution is

$$y(t)=K_1\mathrm{e}^{-2t}+K_2\mathrm{e}^{-3t}+2\cos t+\sin t\,.$$

Finally the solution $y(t)$ with initial conditions $\dot{y}(0)=0$ and $y(0)=1$ could be solved as

$$y(t)=\left(-4\mathrm{e}^{-2t}+3\mathrm{e}^{-3t}+2\cos t+\sin t\right)u(t).$$

5.(1)



(2) The characteristic equation is $2s^3+s^2+5s+6=0$ and corresponding roots are -1 and $\dfrac{1\pm j\sqrt{47}}{4}$. So this system is unstable due to two conjugate roots on the right-hand sided plane.

6.(1) $$R_1\left(C\frac{dy(t)}{dt}+\frac{1}{R_2}y(t)\right)+L\frac{d}{dt}\left(C\frac{dy(t)}{dt}+\frac{1}{R_2}y(t)\right)+y(t)=u(t)$$

$$LC\ddot{y}(t)+\left(R_1C+\frac{L}{R_2}\right)\dot{y}(t)+\left(\frac{R_1}{R_2}+1\right)y(t)=u(t)$$

(2) If $L=4\mathrm{H}$, $C=0.25\mathrm{F}$, $R_1=6\Omega$ and $R_2=1\Omega$, the previous equation became

$$\ddot{y}(t)+5.5\dot{y}(t)+7y(t)=1,\quad t>0\,.$$

As the impulse response of an LTI system is the differential of the corresponding unit-step response, we first find out the unit-step response. Assume that

$$y_p(t)=A,\quad \dot{y}_p(t)=\ddot{y}_p(t)=0,$$

and we know that $7A=1$ and then $A=\frac{1}{7}$, $y_p(t)=\frac{1}{7}$. Further let $y_h(t)=Ke^{st}$ and get the characteristic equation

$$s^2+5.5s+7=0,$$

and roots of the equation are -2, -3.5. Adding $y_p(t)$ and $y_h(t)$ up, the candidate of unit-step response is shown as

$$y(t) = K_1 e^{-2t} + K_2 e^{-3.5t} + \frac{1}{7}.$$

Arbitrary designing initial conditions $y(t) = \dot{y}(t) = 0$, coefficients K_1 and K_2 could be determined, $K_1 = -\frac{1}{3}$ and $K_2 = \frac{4}{21}$. According to the preceding argument, the impulse response of an LTI system is the differential of the corresponding unit-step response, we conclude that

$$\begin{aligned} h(t) &= \frac{dy(t)}{dt} \\ &= \frac{d}{dt}\left(-\frac{1}{3}e^{-2t} + \frac{4}{21}e^{-3.5t} + \frac{1}{7}\right) \\ &= \frac{2}{3}\left(e^{-2t} - e^{-3.5t}\right)u(t). \end{aligned}$$

7. First perform time shift with two samples and another form of the difference equation is shown as

$$y[n] + 3y[n-1] + 2y[n-2] = 8.4 \times 5^n.$$

Assume that $y_p[n] = A \times 5^n$, further we know $y_p[n-1] = 0.2A \times 5^n$ and $y_p[n-2] = 0.04A \times 5^n$. After substituting the preceding assumption into the difference equation, we obtained

$$A \times 5^n + 0.6A \times 5^n + 0.08A \times 5^n = 8.4 \times 5^n.$$

$A = 5$ could be known via the coefficients comparison and $y_p[n]$ was also obtained:

$$y_p[n] = 5^{n+1}.$$

Furthermore, set $y_h[n] = Kc^n$ and substitute it into the corresponding homogeneous equation, we could get the characteristic equation of c:

$$c^2 + 3c + 2 = 0.$$

Then the characteristic equation was solved and known that $c = -1, -2$, so the solution is

$$y[n] = K_1(-1)^n + K_2(-2)^n + 5^{n+1}.$$

Finally the solution $y[n]$ with initial conditions $y[-2] = 0$ and $y[-1] = 1$ could be solved as

$$y[n] = \left[-0.4(-1)^n + 0.8(-2)^n + 5^{n+1}\right]u[n].$$