1. Calculate the derivatives of $f(t)$ and $g(t)$, where

(1) $f(t)=\int_t^{\infty}\left(\tau\cdot e^{-3\tau}\right)d\tau$ (2) $g(t)=\int_0^{t}(\pi-\sin\tau)d\tau$

Sol:

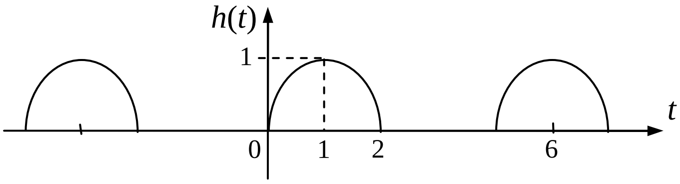
(1) $f(t)=\int_t^{\infty}\left(\tau\cdot e^{-3\tau}\right)d\tau=\int_0^{\infty}\left(\tau\cdot e^{-3\tau}\right)d\tau-\int_o^{t}\left(\tau\cdot e^{-3\tau}\right)d\tau$

Hence, $f'(t)=-t\cdot e^{-3t}$

(2) $g'(t)=t^2-\sin t+\int_0^{t}\frac{\partial}{\partial t}(\pi-\sin\tau)d\tau$

$$=t^2-\sin t+\int_0^{t}\tau d\tau=t^2-\sin t+\frac{1}{2}t^2=\frac{3}{2}t^2-\sin t$$

2. Find the Fourier series of the periodic function $h(t)$ in terms of A_0, A_k and B_k. Note that $h(t)=\sin(\omega t)$ for $0\le t\le 2$.



Sol:

From the figure, the period of $h(t)$ is T=5 and $h(t)=\sin(\pi t/2)$ for $0\le t\le 2$. We have

$$A_0=\frac{1}{5}\int_0^{2}\sin\left(\frac{\pi}{2}t\right)dt=-\frac{2}{5\pi}\cos\frac{\pi}{2}t\Big|_{t=0}^{2}=\frac{4}{5\pi}$$

$$A_k=\frac{2}{5}\int_0^{2}\sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{2k\pi}{5}t\right)dt=\frac{1}{5}\int_0^{2}\left[\sin\left(\frac{\pi}{2}+\frac{2k\pi}{5}\right)t+\sin\left(\frac{\pi}{2}-\frac{2k\pi}{5}\right)t\right]dt$$

$$=\frac{1}{5}\int_0^{2}\left[\sin\frac{(5+4k)\pi}{10}t+\sin\frac{(5-4k)\pi}{10}t\right]dt$$

$$=\frac{1}{5}\left(\frac{-10}{(5+4k)\pi}\cos\frac{(5+4k)\pi}{10}t\Big|_{t=0}^{2}+\frac{-10}{(5-4k)\pi}\cos\frac{(5-4k)\pi}{10}t\Big|_{t=0}^{2}\right)$$

$$=\frac{2}{(5+4k)\pi}\left(1-\cos\left(\pi+\frac{4k}{5}\pi\right)\right)+\frac{2}{(5-4k)\pi}\left(1-\cos\left(\pi-\frac{4k}{5}\pi\right)\right)$$

$$=\frac{2}{(5+4k)\pi}\left(1+\cos\left(\frac{4k}{5}\pi\right)\right)+\frac{2}{(5-4k)\pi}\left(1+\cos\left(\frac{4k}{5}\pi\right)\right)$$

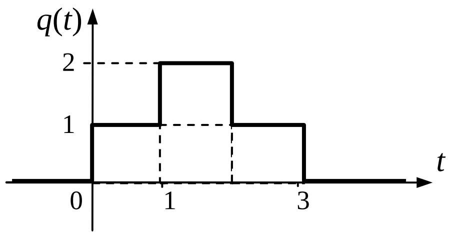
$$=\frac{20}{(5+4k)(5-4k)\pi}\left(1+\cos\left(\frac{4k}{5}\pi\right)\right)$$

$$
\begin{aligned}
B_k &= \frac{2}{5}\int_0^2 sin\left(\frac{\pi}{2}t\right)sin\left(\frac{2k\pi}{5}t\right)dt = \frac{1}{5}\int_0^2\left[cos\left(\frac{\pi}{2}-\frac{2k\pi}{5}\right)t - cos\left(\frac{\pi}{2}+\frac{2k\pi}{5}\right)t\right]dt \\
&= \frac{1}{5}\int_0^2\left[cos\frac{(5-4k)\pi}{10}t - cos\frac{(5+4k)\pi}{10}t\right]dt \\
&= \frac{1}{5}\left(\frac{10}{(5-4k)\pi}sin\frac{(5-4k)\pi}{10}t\Big|_{t=0}^{2} - \frac{10}{(5+4k)\pi}sin\frac{(5+4k)\pi}{10}t\Big|_{t=0}^{2}\right) \\
&= \frac{2}{(5-4k)\pi}sin\left(\pi-\frac{4k}{5}\pi\right) - \frac{2}{(5+4k)\pi}sin\left(\pi+\frac{4k}{5}\pi\right) \\
&= \frac{2}{(5-4k)\pi}sin\left(\frac{4k}{5}\pi\right) + \frac{2}{(5+4k)\pi}sin\left(\frac{4k}{5}\pi\right) \\
&= \frac{20}{(5-4k)(5+4k)\pi}sin\left(\frac{4k}{5}\pi\right)
\end{aligned}
$$

Therefore,

$$
h(t) = A_0 + \sum_{k=1}^{\infty} A_k\, cos\frac{2k\pi}{5}t + \sum_{k=1}^{\infty} B_k\, sin\frac{2k\pi}{5}t\ .
$$

3. Find the Fourier transform of the finite duration function $q(t)$ shown below.
   [Hint: The function q(t) is composed of three pulses, $p(t)$, $2p(t-1)$ and $p(t-2)$.]



Sol:

From the figure, the function $q(t)$ is composed of three pulse $q_1(t)$, $q_2(t)$ and $q_3(t)$, where $q_2(t)=2\,q_1(t-1)$ and $q_3(t)= q_1(t-2)$. Let $Q(\omega)$ and $Q_1(\omega)$ be the transfer functions of $q(t)$ and $q_1(t)$, then $Q(\omega)=Q_1(\omega)(1+2e^{-j\omega}+ e^{-j2\omega})$. First, we calculate the Fourier transform of $Q_1(\omega)$ as below:

$$
Q_1(\omega) = \int_0^1 e^{-j\omega t}\, d\,t = \frac{1}{-\,j\omega}e^{-j\omega t}\Big|_{t=0}^{1} = \frac{j}{\omega}\left(e^{-j\omega}-1\right)
$$

Hence, the Fourier transform of $Q(\omega)$ is

$$
\begin{aligned}
Q(\omega) &= Q_1(\omega)\left(1+2e^{-j\omega}+e^{-j2\omega}\right) = \frac{j}{\omega}\left(e^{-j\omega}-1\right)\left(1+2e^{-j\omega}+e^{-j2\omega}\right) \\
&= \frac{j}{\omega}\left(-1-e^{-j\omega}+e^{-2j\omega}+e^{-3j\omega}\right)
\end{aligned}
$$

4. Consider $f(t)$ and its Laplace transform $F(s) = \dfrac{as^3 + s + 10}{s^4 + 5s^3 + 9s^2 + 5s + k}$.

Determine $f(t)$ for $t \geq 0$ if $f(0)=0$ and $f(\infty)$ is a nonzero constant.

Sol:

According to the initial value theorem, we have

$$f(0) = \lim_{s\to\infty} sF(s) = \left.\frac{as^4 + s^2 + 10s}{s^4 + 5s^3 + 9s^2 + 5s + k}\right|_{s\to\infty} = a = 0$$

According to the final value theorem, we have

$$f(\infty) = \lim_{s\to 0} sF(s) = \left.\frac{as^4 + s^2 + 10s}{s^4 + 5s^3 + 9s^2 + 5s + k}\right|_{s\to 0} = \frac{0}{k} \neq 0$$

which implies $k=0$. Therefore, $F(s) = \dfrac{s+10}{s^4 + 5s^3 + 9s^2 + 5s}$ and then

$$F(s) = \frac{s+10}{s^4 + 5s^3 + 9s^2 + 5s} = \frac{s+10}{s(s+1)(s^2 + 4s^2 + 5)}$$
$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C(s+2)+D}{(s+2)^2+1}$$

which leads to

$$s + 10 = A(s+1)\left((s+2)^2+1\right) + Bs\left((s+2)^2+1\right) + Cs(s+1)(s+2) + Ds(s+1)$$

(1) $s=0 \Rightarrow\ 10 = 5A \ \Rightarrow\ A = 2$

(2) $s=-1 \Rightarrow\ 9 = -2B \ \Rightarrow\ B = -4.5$

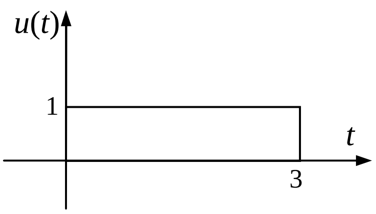
(3) $s=-2 \Rightarrow\ 8 = -A - 2B + 2D \ \Rightarrow\ D = 0.5$

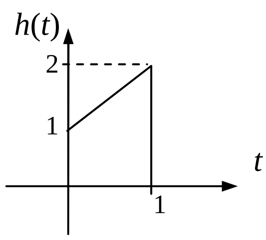
(4) $s=1 \Rightarrow\ 11 = 20A + 10B + 6C + 2D \ \Rightarrow\ C = 2.5$

Hence, we have

$$f(t) = 2 - 4.5e^{-t} + 2.5e^{-2t}\cos(t) + 0.5e^{-2t}\sin(t)$$

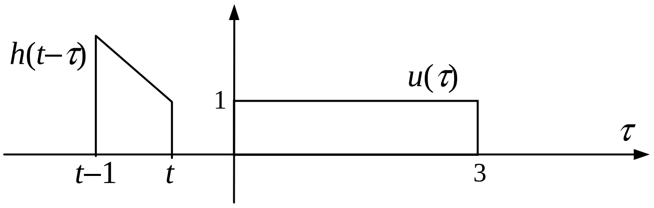
5. Find the convolution $y(t) = u(t) * h(t)$ where $u(t)$ and $h(t)$ are depicted below:



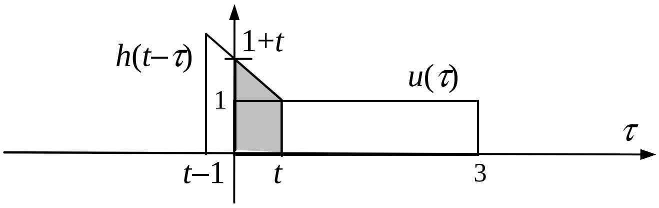


Sol:

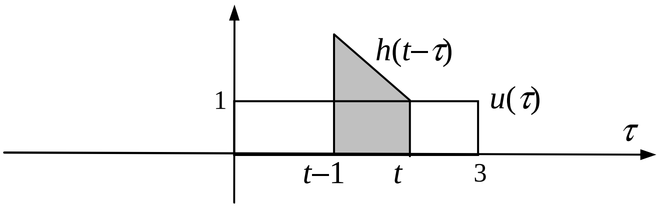
(1) For $t<0$, $y(t) = u(t) * h(t) = \int_{\tau=-\infty}^{t} h(t-\tau)u(\tau)d\tau = 0$.



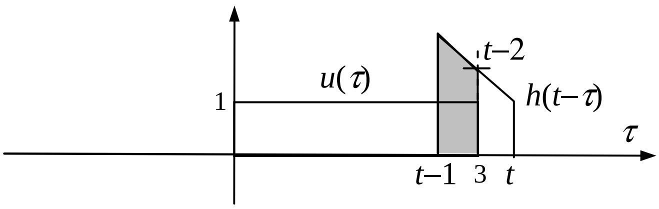
(2) For 0≤t<1, $y(t)=u(t)*h(t)=\int_{\tau=0}^{t}h(t-\tau)u(\tau)d\tau=\frac{[(1+t)+1]t}{2}=\frac{1}{2}t(t+2)$.



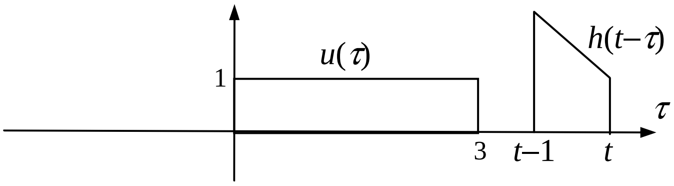
(3) For 1≤t<3, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{t}h(t-\tau)u(\tau)d\tau=\frac{1}{2}1\cdot(1+2)=\frac{3}{2}$.



(4) For 3≤t≤4, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{3}h(t-\tau)u(\tau)d\tau=\frac{[(t-2)+2](4-t)}{2}=\frac{1}{2}t(4-t)$.



(5) For t>4, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{t}h(t-\tau)u(\tau)d\tau=0$.



Hence,

$$y(t)=u(t)*h(t)=\begin{cases}0 & t<0\\ 0.5t(t+2) & 0\le t<1\\ 1.5 & 1\le t<3\\ 0.5t(4-t) & 3\le t\le 4\\ 0 & t>4\end{cases}$$

6. Consider the following system:

$$\ddot{y}(t)+4\dot{y}(t)+8y(t)=\dot{u}(t)+u(t)$$

(1) What is the transfer function $H(s)$?

(2) If $u(t)=cos(3t)-3sin(5t)$, what is the output $y(t)$ as $t\to\infty$?

(3) Under the initial conditions $y(0)=1$ and $\dot{y}(0)=-1$, if $u(t)=3e^{-t}$ then what is $y(t)$ for $t>0$?

[You can solve it by taking Laplace transform, but don't forget $u(0)$.]

<u>Sol:</u>

Taking the Laplace transform yields

$$s^2Y(s)-sy(0)-\dot{y}(0)+4sY(s)-4y(0)+8Y(s)=sU(s)-u(0)+U(s)$$

and thus $Y(s)=\dfrac{s+1}{s^2+4s+8}U(s)+\dfrac{sy(0)+\dot{y}(0)+4y(0)-u(0)}{s^2+4s+8}$

(1) By neglecting all the initial conditions, we have the transfer function

$$H(s)=\frac{Y(s)}{U(s)}=\frac{s+1}{s^2+4s+8}$$

(2) Since the input is $u(t)=cos(3t)-3sin(5t)$, we have the output expressed as

$$y(t)=|H(j3)|cos(3t+\angle H(j3))-3|H(j5)|sin(5t+\angle H(j5))$$

where $H(j3)=\dfrac{j3+1}{-1+j12}=0.263e^{-j23.2^\circ}$ and $H(j5)=\dfrac{j5+1}{-17+j20}=0.194e^{-j51.7^\circ}$

That means

$$y(t)=0.263cos(3t-23.2^\circ)-0.583sin(t-51.7^\circ)$$

(3) Since $s^2Y(s)-sy(0)-\dot{y}(0)+4sY(s)-4y(0)+8Y(s)=sU(s)-u(0)+U(s)$,

we have $s^2Y(s)-s+1+4sY(s)-4+8Y(s)=\dfrac{3s}{s+1}-3+\dfrac{3}{s+1}=0$, i.e.,

$$Y(s)=\frac{s+3}{s^2+4s+8}=\frac{(s+2)+0.5\times 2}{(s+2)^2+2^2}$$

Hence, $y(t)=e^{-2t}\cos 2t+0.5e^{-2t}\sin 2t$.

7. Consider the following first order lowpass filter:

$$\dot{y}(t)+4y(t)=8u(t)$$

(1) Roughly draw the Bode plot of the filter, i.e. the magnitude plot

$|H(j\omega)|_{dB}$ vs. $log\,\omega$ and the phase plot $\angle H(j\omega)$ vs. $log\,\omega$.

(2) Calculate the frequency of half-power point, i.e., the frequency ω satisfies

$|H(j\omega)|=\dfrac{1}{\sqrt{2}}$ or $|H(j\omega)|_{dB}=-3$.

Sol:

By neglecting the initial conditions, take the Laplace transform as below:

$$sY(s)+4Y(s)=8U(s)$$

We have $H(s)=\dfrac{Y(s)}{U(s)}=\dfrac{8}{s+4}=\dfrac{2}{1+\dfrac{s}{4}}$ and $H(j\omega)=\dfrac{2}{1+j\dfrac{\omega}{4}}$

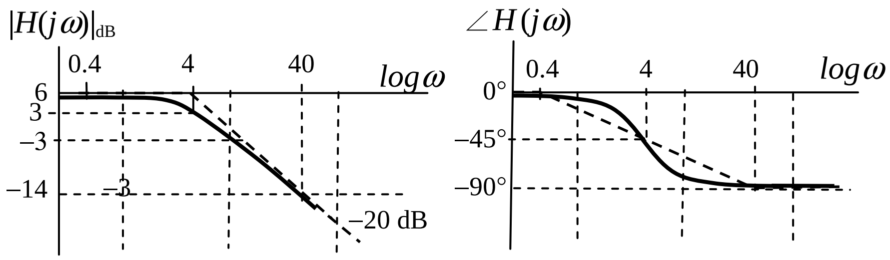
Then,

$$\left|H(j\omega)\right|_{dB}=20\,log\left|H(j\omega)\right|=20\,log\,2-10\,log\left(1+\left(\frac{\omega}{4}\right)^2\right)$$

$$=6.02-10\,log\left(1+\left(\frac{\omega}{4}\right)^2\right)$$

$$\angle H(j\omega)=-tan^{-1}\left(\frac{\omega}{4}\right)$$

(1) The Bode plot is given as below:



(2) Since $\left|H(j\omega)\right|_{dB}=6.02-10\,log\left(1+\left(\frac{\omega}{4}\right)^2\right)$, the frequency ω of half power satisfies $6.02-10\,log\left(1+\left(\frac{\omega}{4}\right)^2\right)=-3$, i.e.,

$$log\left(1+\left(\frac{\omega}{4}\right)^2\right)=0.902 \;\Rightarrow\; 1+\left(\frac{\omega}{4}\right)^2=10^{0.902}=7.98 \;\Rightarrow \omega=10.57\text{ rad}$$

From the Bode plot, the halp power point is around ω=10 rad.