1. Consider a third-order filter given as $\dddot{y}(t)+4\ddot{y}(t)+a\dot{y}(t)+10y(t)=2\dot{u}(t)-u(t)$,

8% where $u(t)$ and $y(t)$ are the input and output.

8% (1) If the system has an eigenvalue –2, then what is the impulse response?

(2) If $u(t)=cos(3t)$, than what is $y(t)$ as $t\to\infty$?

***Sol:***

(1) Since the characteristic equation is $\lambda^3+4\lambda^2+a\lambda+10=0$, we have

$$(-2)^3+4(-2)^2+a(-2)+10=0 \Rightarrow a=9$$

Then, the transfer function is $H(s)=\dfrac{2s-1}{s^3+4s^2+9s+10}$.

By partial fraction expansion, we obtain $H(s)=\dfrac{-1}{s+2}+\dfrac{(s+1)+2\cdot(0.5)}{(s+1)^2+2^2}$

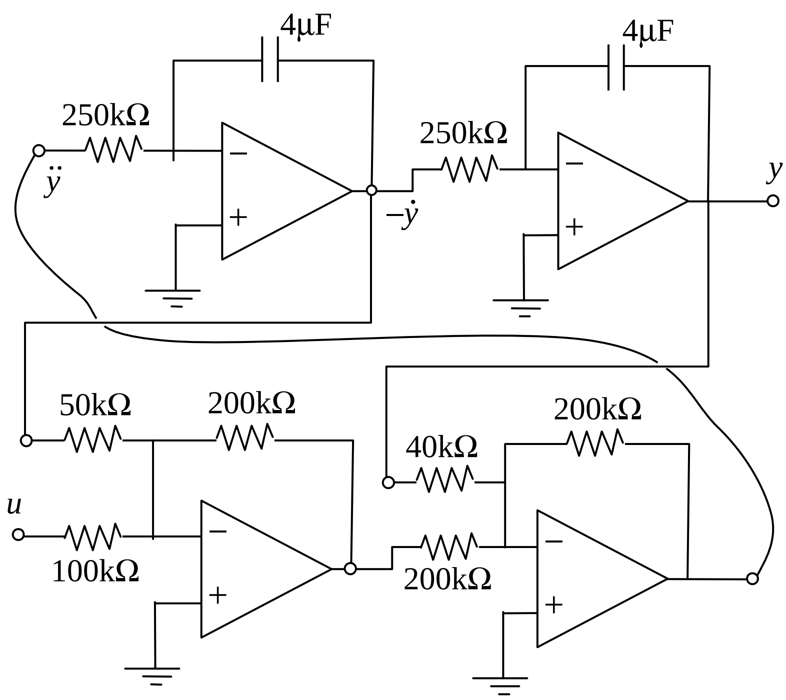
and thus the impulse response $h(t)=-e^{-2t}+e^{-t}(cos\,2t+0.5\,sin\,2t)$.

(2) Since $H(j3)=\dfrac{1-j6}{26}=0.234e^{-80.54^\circ}$, we have $y(t)=0.234\,cos(3t-80.54^\circ)$.

2. A filter in I/O description is designed as $\ddot{y}(t)+4\dot{y}(t)+5y(t)=2u(t)$, where $u(t)$

12% and $y(t)$ are the system input and output. Please realize it by the use of resistors $R$, capacitors $C$ and ideal OpAmps. (Show the values of $R$ and $C$ in your answer.)

***Sol:***



3. A third-order system is expressed in state-space description as below:

15%

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ -6 & -4 & -3\end{bmatrix}\cdot\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\cdot u(t)$$

$$y(t) = x_1(t) - x_2(t)$$

Transfer it into the I/O description as below:

$$\dddot{y}(t) + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0 u(t).$$

***Sol:***

The state equation is $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$ and the output equation is $\boldsymbol{y}(t) = \boldsymbol{c}\boldsymbol{x}(t)$, where

$$\boldsymbol{A} = \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ -6 & -4 & -3\end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix},\ \boldsymbol{c} = [1 \quad -1 \quad 0]$$

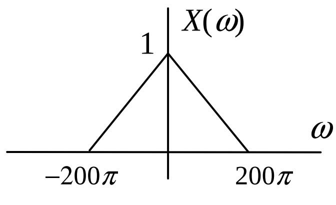
The characteristic polynomial is $|\lambda\boldsymbol{I} - \boldsymbol{A}| = \begin{bmatrix}\lambda & 0 & -1\\ -1 & \lambda & 0\\ 6 & 4 & \lambda+3\end{bmatrix} = \lambda^3 + 3\lambda^2 + 6\lambda + 4$

which implies $\boldsymbol{A}^3 + 3\boldsymbol{A}^2 + 6\boldsymbol{A} + 4\boldsymbol{I} = \boldsymbol{0}$. Therefore, we have

$$\begin{aligned}
&\dddot{y}(t) + 3\ddot{y}(t) + 6\dot{y}(t) + 4y(t)\\
&= \boldsymbol{c}(\dddot{\boldsymbol{x}}(t) + 3\ddot{\boldsymbol{x}}(t) + 6\dot{\boldsymbol{x}}(t) + 4\boldsymbol{x}(t))\\
&= \boldsymbol{c}(\boldsymbol{A}^3 + 3\boldsymbol{A}^2 + 6\boldsymbol{A} + 4)\boldsymbol{x}(t)\\
&\qquad + \boldsymbol{cb}\ddot{u}(t) + (\boldsymbol{cAb} + 3\boldsymbol{cb})\dot{u}(t) + (\boldsymbol{cA}^2\boldsymbol{b} + 3\boldsymbol{cAb} + 6\boldsymbol{cb})u(t)\\
&= \boldsymbol{cb}\ddot{u}(t) + (\boldsymbol{cAb} + 3\boldsymbol{cb})\dot{u}(t) + (\boldsymbol{cA}^2\boldsymbol{b} + 3\boldsymbol{cAb} + 6\boldsymbol{cb})u(t)\\
&= \ddot{u}(t) + 3\dot{u}(t) - 4u(t)
\end{aligned}$$

Hence, $\dddot{y}(t) + 3\ddot{y}(t) + 6\dot{y}(t) + 4y(t) = \ddot{u}(t) + 3\dot{u}(t) - 4u(t)$.

4. The frequency spectrum of $x(t)$ is shown on the right.




4%

8%

Let $x_s(t) = x(t)\delta_T(t)$ where $\delta_T(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT)$.

(1) What is the maximum sampling time to recover $x(t)$ from $x_s(t)$?

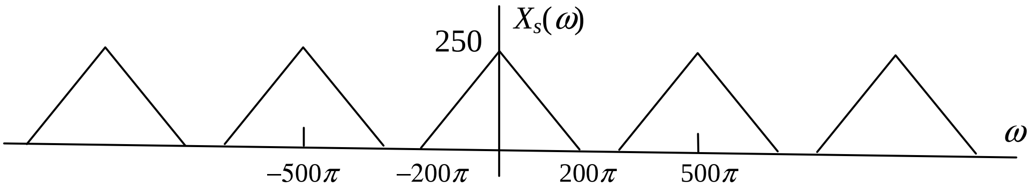
(2) Draw the frequency spectrum of $x_s(t)$ for sampling time $T$=0.004 sec.

***Sol:***

(1) Since the Nyquist rate is $400\pi$, the maximum sampling time is

$$T_m = \frac{2\pi}{400\pi} = 0.005\,\text{sec}.$$

(2) The sampling time $T$=0.004 sec leads to $\omega_s = \dfrac{2\pi}{0.004} = 500\pi$. The frequency spectrum is then given as below:




5. A continuous system is described as $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$ and it is known that

15% $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)d\tau$. Let $u(t) = u(kT)$ for $kT \le t < (k+1)T$, where $T$ is the sampling time. Please derive the discrete system $\boldsymbol{x}[k+1] = \boldsymbol{A}_T\boldsymbol{x}[k] + \boldsymbol{b}_T u[k]$, where $\boldsymbol{x}[k] = \boldsymbol{x}(kT)$ and $u[k] = u(kT)$. What are $\boldsymbol{A}_T$ and $\boldsymbol{b}_T$?

***<u>Sol:</u>***

$$\boldsymbol{x}((k+1)T) = e^{(k+1)T\boldsymbol{A}}\boldsymbol{x}(0) + \int_0^{(k+1)T} e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}u(\tau)d\tau$$

$$= e^{T\boldsymbol{A}}\left(e^{kT\boldsymbol{A}}\boldsymbol{x}(0) + \int_0^{(k+1)T} e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(\tau)d\tau\right)$$

$$= e^{T\boldsymbol{A}}\left(e^{kT\boldsymbol{A}}\boldsymbol{x}(0) + \int_0^{kT} e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(\tau)d\tau + \int_{kT}^{(k+1)T} e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(\tau)d\tau\right)$$

$$= e^{T\boldsymbol{A}}\left(\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} e^{\boldsymbol{A}(kT-\tau)}\boldsymbol{b}u(kT)d\tau\right)$$

$$= e^{T\boldsymbol{A}}\boldsymbol{x}(kT) + \left(\int_{kT}^{(k+1)T} e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}d\tau\right)u(kT)$$

Define $\boldsymbol{x}[k] = \boldsymbol{x}(kT)$ and $u[k] = u(kT)$, then $\boldsymbol{x}[k+1] = \boldsymbol{A}_T\boldsymbol{x}[k] + \boldsymbol{b}_T u[k]$

where $\boldsymbol{A}_T = e^{T\boldsymbol{A}}$ and $\boldsymbol{b}_T = \int_{kT}^{(k+1)T} e^{\boldsymbol{A}((k+1)T-\tau)}\boldsymbol{b}d\tau$.

6. Assume z is in the region of convergence for the following z-transforms.

6% (1) Determine $x[0]$, $x[1]$, $x[2]$ and $x[3]$ for $X(z) = \dfrac{z^3 + 0.5z^2}{z^4 + 0.9z^3 + 0.6z^2 - z + 1}$.

9% (2) Find $x[k]$, $k \ge 0$, for $X(z) = \dfrac{z}{(z-0.6)(z^2 - z + 1)}$.

***Sol:***

(1) $X(z)=\dfrac{z^3+0.5z^2}{z^4+0.9z^3+0.6z^2-z+1}=\dfrac{z^{-1}+0.5z^{-2}}{1+0.9z^{-1}+0.6z^{-2}-z^{-3}+z^{-4}}$

$$
\begin{array}{r|l}
 & z^{-1}-0.4z^{-2}-0.24z^{-3}+\ldots\ldots\ldots\ldots \\
1+0.9z^{-1}+0.6z^{-2}-z^{-3}+z^{-4} & z^{-1}+0.5z^{-2} \\
 & z^{-1}+0.9z^{-2}+0.60z^{-3}-1.00z^{-4}+1.0z^{-5} \\ \hline
 & -0.4z^{-2}-0.60z^{-3}+1.00z^{-4}-1.0z^{-5} \\
 & -0.4z^{-2}-0.36z^{-3}-0.24z^{-4}+0.4z^{-5}-0.4z^{-6} \\ \hline
 & -0.24z^{-3}+\ldots\ldots\ldots\ldots
\end{array}
.
$$

Hence, $x[0]=0$, $x[1]=1$, $x[2]=-0.4$ and $x[3]=-0.24$.

(2) $X(z)=\dfrac{z}{(z-0.6)(z^2-z+1)}\Rightarrow\dfrac{X(z)}{z}=\dfrac{1}{(z-0.6)(z^2-z+1)}$

$$\Rightarrow\frac{X(z)}{z}=\frac{1.316}{z-0.6}+\frac{-1.316(z-0.5)-0.152\times0.866}{z^2-z+1}$$

$$\Rightarrow X(z)=\frac{1.316z}{z-0.6}+\frac{-1.316(z^2-0.5z)-0.152\times(0.866z)}{z^2-z+1}$$

Hence, $x[k]=1.316\times(0.6)^k-1.316\times cos\left(\dfrac{\pi}{3}k\right)-0.152\times sin\left(\dfrac{\pi}{3}k\right)$.

| 5% |
| --- |
| 10% |

7. Consider the discrete system: $y[k]+ay[k-1]+by[k-2]=0.4u[k-2]$, where $y[k]=0$ for $k<0$. If the roots of its characteristic eqaution are 0.7 and 0.9.

(1) What is the transfer function? (2) If $u[k]=0.5$ for $k\geq0$, what is $y[k]$?

***Sol:***

(1) Sincce $(\lambda-0.7)(\lambda-0.9)=\lambda^2-1.6\lambda+0.63$, we have

$$H(z)=\frac{0.4}{(z-0.7)(z-0.9)}=\frac{0.4}{z^2-1.6z+0.63}$$

(2) Since the z-transform of $u[k]=0.5$ is $U(z)=\dfrac{z}{z-1}$, we have

$$Y[z]=H(z)U(z)=\frac{0.4z}{(z-0.7)(z-0.9)(z-1)}\quad\text{and then}$$

$$\frac{Y[z]}{z}=\frac{0.4}{(z-0.7)(z-0.9)(z-1)}=\frac{6.67}{z-0.7}-\frac{20}{z-0.9}+\frac{13.33}{z-1}$$

Therefore, $y[k]=6.67\times(0.7)^k-20\times(0.9)^k+13.33$.