1. Find the convolution $y(t)=u(t)*h(t)$ where $u(t)$ and $h(t)$ are depicted below:

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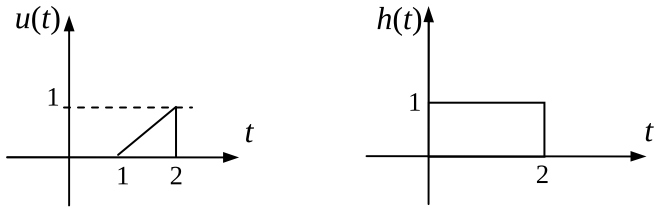
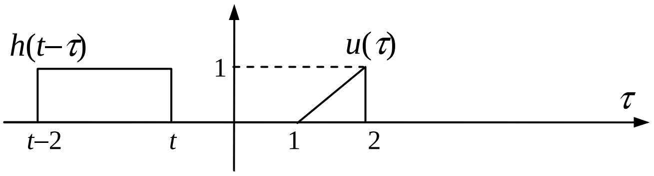
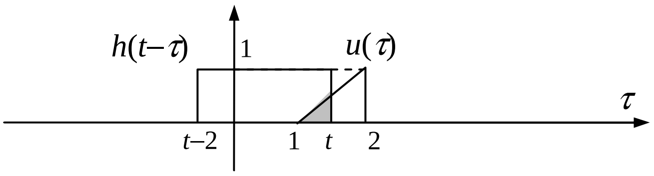


Sol:

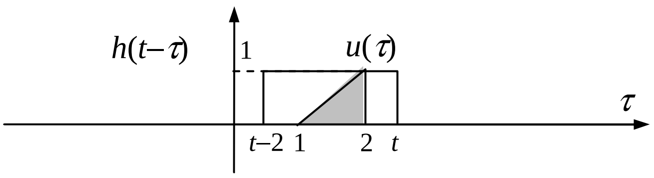
(1) For $t<1$, $y(t)=u(t)*h(t)=\int_{\tau=-\infty}^{t} h(t-\tau)u(\tau)d\tau=0$.



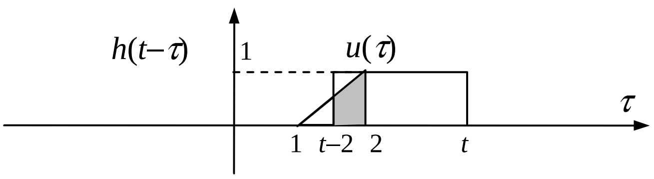
(2) For $1\le t<2$, $y(t)=u(t)*h(t)=\int_{\tau=0}^{t} h(t-\tau)u(\tau)d\tau=0.5(t-1)^2$.



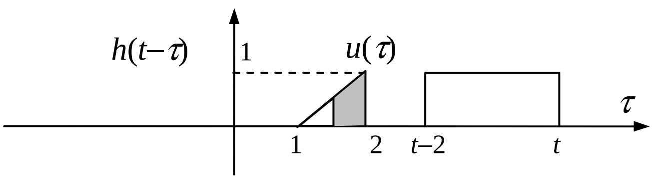
(3) For $2\le t<3$, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{t} h(t-\tau)u(\tau)d\tau=0.5$.



(4) For $3\le t\le 4$, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{3} h(t-\tau)u(\tau)d\tau=0.5-0.5(t-3)^2$.



(5) For $t>4$, $y(t)=u(t)*h(t)=\int_{\tau=t-1}^{t} h(t-\tau)u(\tau)d\tau=0$.



Hence,

$$y(t) = u(t) * h(t) = \begin{cases} 0 & t < 1 \\ 0.5(t-1)^2 & 1 \le t < 2 \\ 0.5 & 2 \le t < 3 \\ 0.5 - 0.5(t-3)^2 & 3 \le t \le 4 \\ 0 & t > 4 \end{cases}$$

2. Consider a system given as $\dddot{y}(t) + 6\ddot{y}(t) + \alpha\dot{y}(t) + \beta y(t) = \ddot{u}(t) - u(t)$, where $u(t)$ and $y(t)$ are the input and output. It is known that the system has an eigenvalue –2 and its transfer function is $H(s) = \dfrac{s - z_0}{s^2 + a_1 s + a_0}$ due to the pole-zero cancellation.

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(1) If the system is unstable and the zero z_0 is located in the left-half complex plane after pole-zero cancellation, what is its transfer function $H(s)$?

(2) If the system is stable and $u(t)=cos(0.5t)$, then what is $y(t)$ as $t \to \infty$?

Sol:

The original transfer function is $H(s) = \dfrac{(s+1)(s-1)}{s^3 + 6s^2 + \alpha s + \beta}$. Clearly, there are two zeros 1 and –1. Since the system has an eigenvalue –2 and pole-zero cancellation, we know that the characteristic polynomial can be represented by

$$s^3 + 6s^2 + \alpha s + \beta = (s+2)(s+1)(s+0.5\beta)$$

or $$s^3 + 6s^2 + \alpha s + \beta = (s+2)(s-1)(s-0.5\beta).$$

(1) Since the system is unstable, the pole-zero cancellation happens at s=1. Thus, the characteristic polynomial is

$$\begin{aligned} s^3 + 6s^2 + \alpha s + \beta &= (s+2)(s-1)(s-0.5\beta) \\ &= s^3 + (1-0.5\beta)s^2 + (-2-0.5\beta)s + \beta \end{aligned}$$

which results in $1 - 0.5\beta = 6$ and $\alpha = -2 - 0.5\beta$, i.e., $\beta = -10$ and $\alpha = 3$. Hence, the charateristic polynomial is

$$\begin{aligned} H(s) &= \frac{(s+1)(s-1)}{s^3 + 6s^2 + 3s - 10} = \frac{(s+1)(s-1)}{(s+2)(s-1)(s+5)} \\ &= \frac{s+1}{(s+2)(s+5)} = \frac{s+1}{s^2 + 7s + 10} \end{aligned}$$

Although this resulted transfer function seems stable, the system behavior is shown to be unstable due to the existence of unstable pole at s=1.

(2) Since the system is stable, the pole-zero cancellation happens at s=–1. Thus, the characteristic polynomial is

$$s^3+6s^2+\alpha s+\beta=(s+2)(s+1)(s+0.5\beta)$$
$$=s^3+(3+0.5\beta)s^2+(2+1.5\beta)s+\beta$$

which results in $3+0.5\beta=6$ and $\alpha=2+1.5\beta$, i.e., $\beta=6$ and $\alpha=11$.
Hence, the charateristic polynomial is

$$H(s)=\frac{(s+1)(s-1)}{s^3+6s^2+11s+6}=\frac{(s+1)(s-1)}{(s+2)(s+1)(s+3)}$$
$$=\frac{s-1}{(s+2)(s+3)}=\frac{s-1}{s^2+5s+6}=\frac{-3}{s+2}+\frac{4}{s+3}$$

and the impulse function is $h(t)=-3e^{-2t}+4e^{-3t}$.

Since the transfer function is $H(s)=\frac{s-1}{s^2+5s+6}$ and $u(t)=cos(0.5t)$, by letting $s=j0.5$ we have

$$H(j0.5)=\frac{j0.5-1}{(j0.5)^2+5(j0.5)+6}=\frac{-1+j0.5}{5.75+j2.5}$$
$$=\frac{1.118\angle 153.4^\circ}{6.270\angle 23.5^\circ}=0.1783\angle 129.9^\circ=0.1783\angle -230.1^\circ$$

Hence, $y(t)=0.1783cos(0.5t+129.9^\circ)$ or $y(t)=0.1783cos(0.5t-230.1^\circ)$.

3. Consider the discrete function $x[k]=1.2-0.8sin\left(k\frac{\pi}{3}+\frac{\pi}{4}\right)$ for $k\geq 0$.

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(1) What is the z-transform $X(z)$?

(2) Neglect the data of $x[k]$ for $k\geq 5$ and calculate the discrete Fourier transform $X[n]$ for n=0,1,2,3,4.

<u>Sol:</u>

The discrete function can be further expressed as

$$x[k]=1.2-\frac{0.8}{\sqrt{2}}\left(sin\left(k\frac{\pi}{3}\right)+cos\left(k\frac{\pi}{3}\right)\right)$$
$$=1.2-0.5657\left(sin\left(k\frac{\pi}{3}\right)+cos\left(k\frac{\pi}{3}\right)\right)$$

(1) Its z-transform is

$$X(z)=\frac{1.2z}{z-1}-0.5657\left(\frac{0.866z}{z^2-z+1}+\frac{z(z-0.5)}{z^2-z+1}\right)$$
$$=\frac{1.2z}{z-1}-\frac{0.5657z^2+0.2070z}{z^2-z+1}=\frac{0.6343z^3-0.8413z^2+1.4070z}{z^3-2z^2+2z-1}$$

(2) Neglect the data of $x[k]$ for $k\geq 5$ then $x[0]=0.6343$, $x[1]=0.4273$, $x[2]=0.9929$, $x[3]=1.7657$ and $x[4]=1.9727$. Calculate the discrete Fourier transform as below

$$X[0]=\sum_{k=0}^{4}x[k]=5.7929,$$

$$X[1]=\sum_{k=0}^{4}x[k]e^{-jk\frac{2\pi}{5}}=-0.8558+j1.9240$$

$$X[2]=\sum_{k=0}^{4}x[k]e^{-jk\frac{4\pi}{5}}=-0.4549+j0.1734$$

$$X[3]=\sum_{k=0}^{4}x[k]e^{-jk\frac{6\pi}{5}}=-0.4549-j0.1734$$

$$X[4]=\sum_{k=0}^{4}x[k]e^{-jk\frac{8\pi}{5}}=-0.8558-j1.9240$$

4. Based on the unit output feedback control technology, design the Controller $C(s)$ for the Plant $H(s)=\dfrac{s+1}{s^2+s+2}$ to drive the output $y(t)$ to follow the command y_d.

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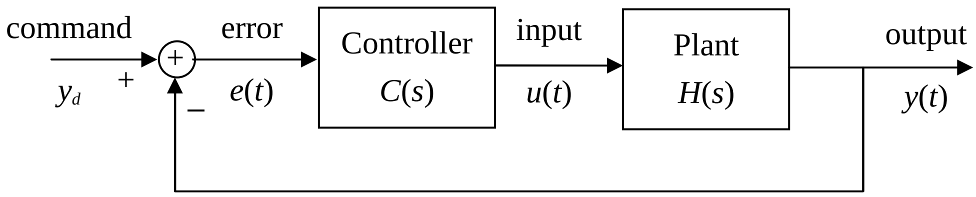
6%



Choose $C(s)=K>0$ where K is constant and set the command as $y_d=1$. Let $Y(s)$ and $Y_d(s)$ be the Laplace transforms of $y(t)$ and y_d, then $Y(s)=H'(s)Y_d(s)$ where $H'(s)$ is the closed-loop transfer function.

(1) What is closed-loop transfer function $H'(s)$ in terms of K?

(2) Determine K if one of the poles of $H'(s)$ is located at $s=-2$.

(3) As $t\to\infty$, determine $y(\infty)$ and the steady-state error $e(\infty)$.

Sol:

(1) closed-loop transfer function $H'(s)$ is

$$H'(s)=\frac{C(s)H(s)}{1+C(s)H(s)}=\frac{K\dfrac{s+1}{s^2+s+2}}{1+K\dfrac{s+1}{s^2+s+2}}=\frac{K(s+1)}{s^2+(K+1)s+K+2}$$

(2) Since one of the poles of $H'(s)$ is $s=-2$, we have

$$s^2+(K+1)s+K+2\Big|_{s=-2}=0\Rightarrow 4-2(K+1)+K+2=0\Rightarrow K=4$$

With $K=4$, we have $H'(s)=\dfrac{4(s+1)}{s^2+5s+6}=\dfrac{4(s+1)}{(s+2)(s+3)}$. Clearly, the poles of the closed-loop system are located at $s=-2$ and $s=-3$. Hence, the closed-loop system is stable due to the fact that both of its poles are located in the left-half complex plane.

(3) The Laplace transform of $y(t)$ is

$$Y(s) = H'(s)Y_d(s) = \frac{4(s+1)}{s^2+5s+6}\cdot\left(\frac{1}{s}\right) = \frac{4(s+1)}{s(s^2+5s+6)}$$

where $Y_d(s) = \frac{1}{s}$. According to the final value theorem, we have

$$y(\infty) = \lim_{s\to 0} sY(s) = \left.\frac{4(s+1)}{s^2+5s+6}\right|_{s=0} = \frac{2}{3} = 0.6667$$

Hence, the steadyu-state error is

$$e(\infty) = y_d - y(\infty) = 1 - 0.6667 = 0.3333\,.$$

[Complement]

With the use of $C(s) = K_p + \frac{K_I}{s}$, we have

$$H'(s) = \frac{C(s)H(s)}{1+C(s)H(s)} = \frac{\left(K_p + \frac{K_I}{s}\right)\frac{s+1}{s^2+s+2}}{1+\left(K_p + \frac{K_I}{s}\right)\frac{s+1}{s^2+s+2}}$$

$$= \frac{K_p s^2 + (K_p + K_I)s + K_I}{s^3 + (K_p+1)s^2 + (K_p + K_I + 2)s + K_I}$$

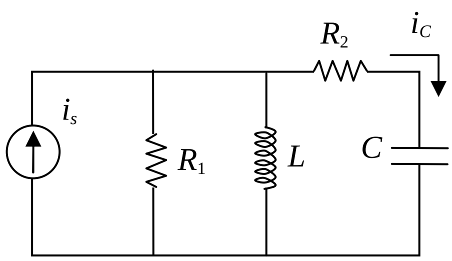
Then, $y(\infty) = \lim_{s\to 0} sY(s) = H'(s)\big|_{s=0} = H'(0) = \frac{K_I}{K_I} = 1$ and thus

$$e(\infty) = y_d - y(\infty) = 1 - 1 = 0$$

The PI control indeed eliminates the steady-state error.

5. Consider the circuit with $R_1 = R_2 = 5\Omega$, $L = 2\text{H}$, and $C = 1/12$ F, if $i_s(t) = -2cos(3t - 45°)$ A,

4%	(1) What is the phasor of $i_s(t)$?
4%	(2) What is the impedance of L and C?
10%	(3) What is $i_C(t)$ as $t\to\infty$?



Sol:

(1) Since $i_s(t) = -2\,cos(3t - 45°) = 2\,cos(3t + 135°)$, the phasor of $i_s(t)$ is

$$I_s = 2e^{j135°} = 2\angle 135° = -1.4142 + j1.4142$$

(2) Since the frequency is $\omega = 3$, the impedance of L and C are

$$Z_L = j\omega L = j6 \quad \text{and} \quad Z_C = \frac{1}{j\omega C} = -j4$$

(3) According to the phasor method, we have

$$I_s = \frac{(R_2 + Z_C)I_C}{R_1} + \frac{(R_2 + Z_C)I_C}{Z_L} + I_C = \left(\frac{5-j4}{5} + \frac{5-j4}{j6} + 1\right)I_C$$
$$= (1.3333 - j1.6333)I_C = 2.1084e^{-j50.8^\circ} I_C$$

Hence, $I_C = \dfrac{I_s}{2.1084e^{-j50.8^\circ}} = \dfrac{2e^{j135^\circ}}{2.1084e^{-j50.8^\circ}} = 0.9486e^{j185.8^\circ}$ which results in $i_C(t) = 0.9486\cos(3t + 185.8^\circ)$

6. Design a 2nd order lowpass digital filter $H(z) = \dfrac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0}$ with sampling rate 200Hz by the use of bilinear transformation such that the low frequency response of $H(z)$ is closed to $H(s) = \dfrac{1}{(1+\alpha s)(1+\beta s)}$. If α=0.05 and β=0.02, determine the coefficients a_1, a_0, b_2, b_1 and b_0 in $H(z)$.

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<u>Sol:</u>

Based on the bilinear transformation, choose $H(p) = \dfrac{1}{(1+0.05p)(1+0.02p)}$

where $p = C\dfrac{1-z^{-1}}{1+z^{-1}}$. Since the sampling rate is 200Hz, the coefficient C is set to be $C = \dfrac{2}{T} = 2\times 200 = 400$. Hence, $p = 400\dfrac{1-z^{-1}}{1+z^{-1}}$ and substitueing it into $H(p)$ yields the following desired digital filter:

$$H(z) = \frac{1}{\left(1 + 0.05\times 400\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)\left(1 + 0.02\times 400\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)}$$

After rearranging, we have

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{189 - 318z^{-1} + 133z^{-2}} = \frac{0.0053z^2 + 0.0106z + 0.0053}{z^2 - 1.6825z + 0.7037}.$$

Hence,

a_1=−1.6825, a_0=0.7037, b_2=0.0053, b_1=0.0106 and b_0=0.0053.