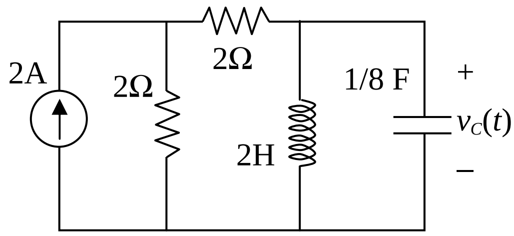
To solve the following circuits, you can use all the methods you learned in this course.

1. (15%) Consider the circuit on the right, where the initial capacitor voltage is $v_C(0)=1\,\text{V}$ and the initial inductor current is $i_L(0)=-1\,\text{A}$. What is the capcitor voltage $v_C(t)$ for $t\geq 0$?

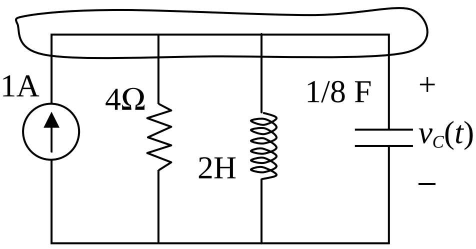



Sol:

The circuit can be transformed into Norton equivalent circuit on the right.

Based on KCL, we have

$$\frac{v_C(t)}{4}+i_L(t)+\frac{1}{8}\dot{v}_C(t)=1$$




Differentiating it yields

$$\frac{\dot{v}_C(t)}{4}+\frac{di_L(t)}{dt}+\frac{1}{8}\ddot{v}_C(t)=0$$

where $\frac{di_L(t)}{dt}=\frac{1}{2}v_L(t)=\frac{1}{2}v_C(t)$.

Hence, $\ddot{v}_C(t)+2\dot{v}_C(t)+4v_C(t)=0$ whose characteristic equation is $\lambda^2+2\lambda+4=0$.

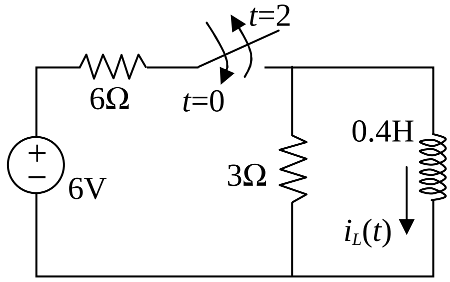
Then, the eigenvalues are $\lambda=-1\pm j\sqrt{3}$. Therefore, $v_C(t)=e^{-t}\left(A_1\cos\sqrt{3}t+A_2\sin\sqrt{3}t\right)$

and $\dot{v}_C(t)=-e^{-t}\left(A_1\cos\sqrt{3}t+A_2\sin\sqrt{3}t\right)+e^{-t}\sqrt{3}\left(-A_1\sin\sqrt{3}t+A_2\cos\sqrt{3}t\right)$.

Since, $v_C(0)=1=A_1$ and $\dot{v}_C(0)=8-\left(2v_C(0)+8i_L(0)\right)=14=-A_1+\sqrt{3}A_2$, we have

$A_1=1$ and $A_2=5\sqrt{3}$, i.e., $v_C(t)=e^{-t}\left(\cos\sqrt{3}t+5\sqrt{3}\sin\sqrt{3}t\right)$ V

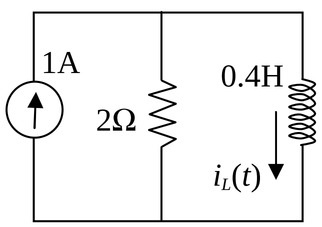
2. (15%) Consider the *RL* circuit on the right, whose switch has been opened for a long time before $t$=0. If the switch is closed at $t$=0 and then opened again at $t$=2, please solve $i_L(t)$ for $0<t<\infty$?




Sol:

At $t$=0, we have $i_L(0)=0$.

For 0<t<2, the Norton equivalent circuit is shown below and described by



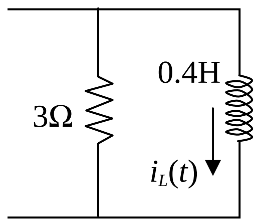

$$\frac{0.4}{2}\frac{di_L(t)}{dt}+i_L(t)=1$$

i.e., $\frac{di_L(t)}{dt}+5i_L(t)=5$.

We have $i_L(t)=-e^{-5t}+1$.

At $t$=2, we have $i_L(2)=-e^{-10}+1$.

For $t$>2, the Norton equivalent circuit is shown below and described by




$$\frac{0.4}{3}\frac{di_L(t)}{dt}+i_L(t)=0$$

i.e., $\frac{di_L(t)}{dt}+7.5i_L(t)=0$.

We have $i_L(t)=\left(1-e^{-10}\right)e^{-5(t-2)}$.

Hence, the inductor current is

$$i_L(t)=\begin{cases}-e^{-5t}+1 & 0<t\le 2\\ \left(1-e^{-10}\right)e^{-5(t-2)} & 2<t<\infty\end{cases}$$

3. (10%) Find the Laplace transforms $H_i(s)$ of the following functions $h_i(t)$:
   $h_1(t)=e^{-2t}(sin3t-2)$, $\quad h_2(t)=t^2\cdot sin3t$

Sol:

$$H_1(s)=\frac{3}{(s+2)^2+9}-\frac{2}{s+2}=\frac{3(s+2)-2\left(s^2+4s+13\right)}{\left(s^2+4s+13\right)(s+2)}=\frac{-2s^2-5s-20}{s^3+6s^2+21s+26}$$

$$H_2(s)=\frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)=\frac{18s^2-54}{\left(s^2+9\right)^3}$$

4. (10%) Find the functions $h_i(t)$ and $h_i(\infty)$ from the following Laplace transforms $H_i(s)$:
   $H_1(s)=\frac{3}{s\left(s^2+2s+2\right)}$, $\quad H_2(s)=\frac{s-1}{(s+1)\left(s^2+4s+4\right)}$

Sol:

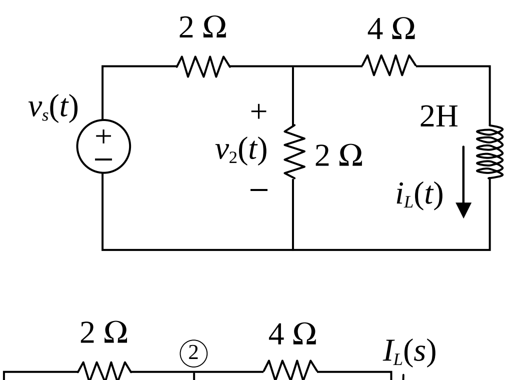
$$H_1(s)=\frac{3}{s\left(s^2+2s+2\right)}=\frac{1.5}{s}-\frac{1.5(s+1)+1.5}{(s+1)^2+1}$$

$\rightarrow\ h_1(t)=1.5-1.5e^{-t}(cost+sint)$ and $h_1(t)=1.5$

$$H_2(s)=\frac{s-1}{(s+1)\left(s^2+4s+4\right)}=\frac{-2}{s+1}+\frac{2}{s+2}+\frac{3}{(s+2)^2}$$

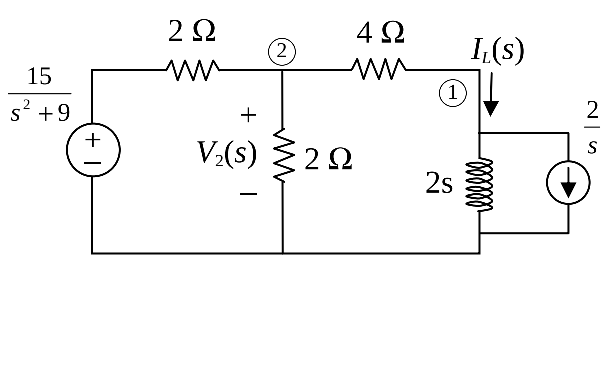
$\rightarrow\ h_2(t)=-2e^{-t}+2e^{-2t}+3te^{-2t}$ and $h_2(t)=0$.

5. (15%) Given the circuit on the right, the initial inductor current is $i_L(0)$=2A and the voltage source is $v_s(t)=5sin3t$ V. What is the voltage $v_2(t)$?




Sol:

The circuit in s-form is on the right.




KCL①: $\frac{V_1-V_2}{4}+\frac{V_1}{2s}+\frac{2}{s}=0$

KCL②: $\frac{V_2-\frac{15}{s^2+9}}{2}+\frac{V_2}{2}+\frac{V_2-V_1}{4}=0$

Hence, $V_2(s) = \dfrac{-2s^2 + 7.5s - 3}{(s^2+9)(s+2.5)} = \dfrac{-2.246}{s+2.5} + \dfrac{0.246s + 3\times 2.295}{s^2+9}$

i.e., $v_2(t) = -2.246e^{-2.5t} + 0.246\cos 3t + 2.295\sin 3t$

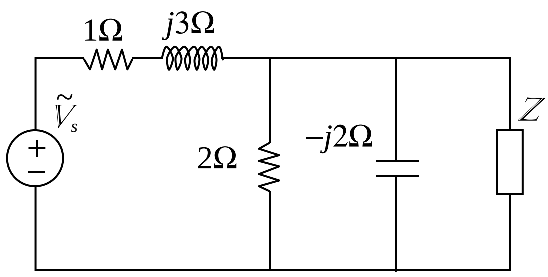
6. (15%) The dynamic equation of a circuit can be described as $V_L(s) = H(s)I_s(s)$, where $I_s(s)$ and $V_L(s)$ are the Laplace transforms of input $i_s(t)$ and output $v_L(t)$. Let the transfer function $H(s) = \dfrac{2s+1}{s^2+5s+3}$ and the input $i_s(t) = 4\cos(\omega t - 20°)$ A with $\omega$=3 rad.
   (A) What is $\mathcal{I}_s$, the phasor of $i_s(t)$?
   (B) If $H(j\omega) = |H(j\omega)|e^{j\theta}$, what are $|H(j\omega)|$ and $\theta$?
   (C) What is the output $v_L(t)$ as $t\to\infty$?

Sol:
(A) $\mathcal{I}_s = 4\angle(-20°) = 4\cos 20° - j4\sin 20° = 3.7588 - j1.3681$

(B) $H(j3) = \dfrac{j6+1}{-9+j15+3} = \dfrac{1+j6}{-6+j15} = 0.3218 - j0.1954 = 0.3765\angle(-31.26°)$

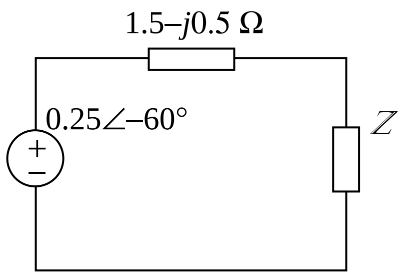
(C) $\mathcal{V}_L = 0.3765\angle(-31.26°)\times 4\angle(-20°) = 1.5060\angle(-51.26°)$
Hence, $v_L(t) = 1.5060\cos(3t - 51.26°)$

7. (20%) Consider the circuit on the right with $\tilde{\mathcal{V}}_s = 5\angle 30°$. If $\mathcal{Z} = R + jX$, determine the maximum power $P_{max}$ transferred to the $\mathcal{Z}$. If $\mathcal{Z} = R$, what is the maximum power $P_{max}$ transferred to $R$?




Sol:
The Thevenin equivalent circuit is on the right.
If $\mathcal{Z} = 1.5 + j0.5$, the maximum power is

$$P_{max} = \frac{\tilde{V}_T^2}{4R_T} = \frac{0.25^2}{4\times 1.5} = 0.0104 \text{ W}$$




If $\mathcal{Z} = R = \sqrt{1.5^2 + 0.5^2} = 1.5811$,

the maximum power is

$$P_{max} = \frac{\tilde{V}_T^2}{2(R+R_T)} = \frac{0.25^2}{2(1.5811+1.5)} = 0.0101 \text{ W}$$