20. Implementation of Sinusoidal Oscillator via Phase Plane Method

An ideal sinusoidal oscillator produces a periodic function with amplitude A and angular frequency ω, described as $y(t)=A cos(\omega t+\phi)$, whose mathematic model in can be described by the following differential equation:

$$\ddot{y}(t)+\omega^2 y(t)=0 \qquad (1)$$

with initial conditions $y(0)=y_0$ and $\dot{y}(0)=\dot{y}_0$. It can be found that the amplitude $A=\sqrt{y_0^2+\dot{y}_0^2}$ and the phase $\phi=-tan^{-1}(\dot{y}_0/\omega y_0)$.

In addition to the differntial equation (1), the sinusoidal oscillator can be also represented by the the following state equation:

$$\begin{cases} \dot{x}_1(t)=\omega x_2(t) \\ \dot{x}_2(t)=-\omega x_1(t) \end{cases} \qquad (2)$$

where the state variables are chosen as $x_1(t)=y(t)$ and $x_2(t)=\dot{y}(t)/\omega$. Their initial conditions are then given as $x_1(0)=y_0$, $x_2(0)=\dot{y}_0/\omega$. Hence, the amplitude and the phase are correspondingly $A=\sqrt{x_1^2(0)+\omega^2 x_2^2(0)}$ and $\phi=-tan^{-1}\left(\dfrac{x_2(0)}{x_1(0)}\right)$.

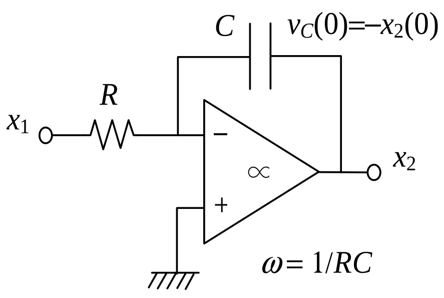

Figure-1

Using operational amplifiers, the equation $\dot{x}_2(t)=-\omega x_1(t)$ in (2) is commonly implemented as the circuit in Figure-1. As for the other equation $\dot{x}_1(t)=\omega x_2(t)$ in (2), it can be implemented by the integrator $\dot{x}_1(t)=-\omega x_3(t)$ cascaded by a negative unit gain $x_3(t)=-x_2(t)$, as shown in Figure-2.

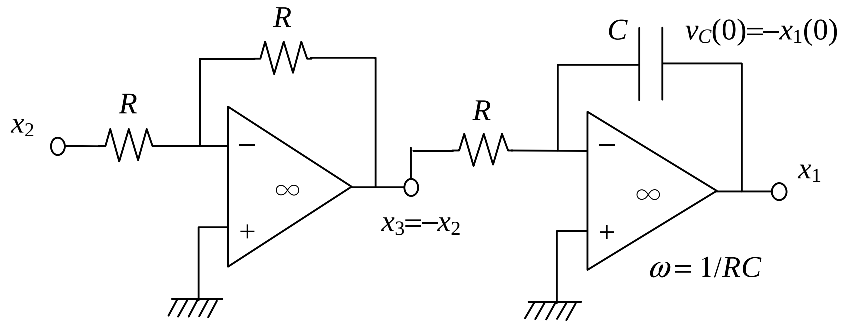

Figure-2

Therefore, the sinusoidal oscillator can be constructed by Figure-1 and Figure-2, illustrated in Figure-3 and the state variable x_1(t) is the desired sinusoidal function $y(t) = Acos(\omega t+\phi)$.

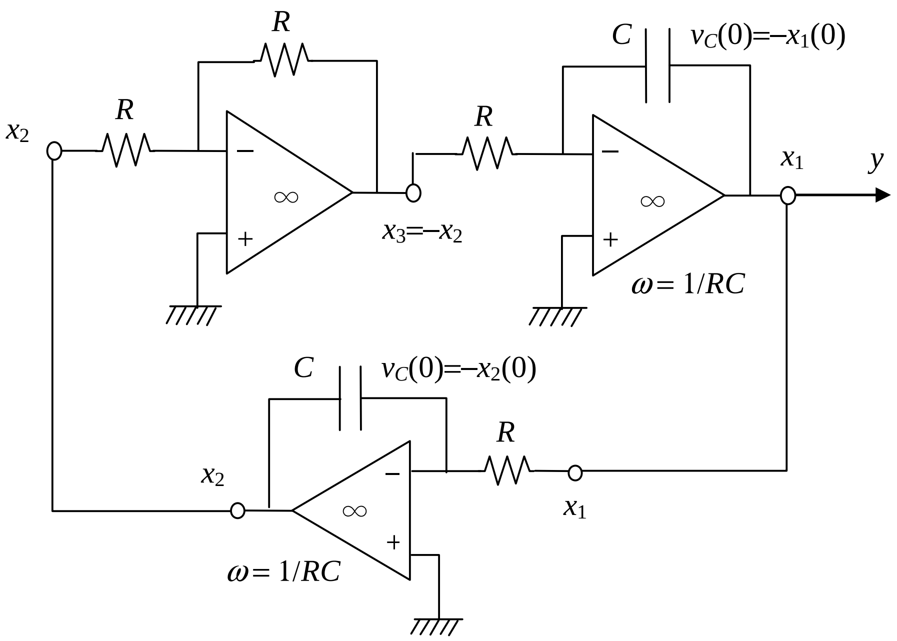

Figure-3

Since $x_1(t) = y(t) = Acos(\omega t+\phi)$ and $x_2(t) = \dot{x}_1(t)/\omega = -Asin(\omega t+\phi)$, it is easy to find that the trajectory follows a circle of $x_1^2(t)+x_2^2(t) = A^2$ with radius A in the phase plane x_1-x_2, depicted in Figure-4. Note that arrows placed on the trajectory show the direction as time t increases. For example, in the first quadrant $x_1(t)>0$ and $x_2(t)>0$, we have $\dot{x}_1(t) = \omega x_2(t) > 0$ and $\dot{x}_2(t) = -\omega x_1(t) < 0$, i.e., $x_1(t)$ is increased and $x_2(t)$ is decreased as time t increases; such phenomenon is drawn in the first quadrant in Figure-4.

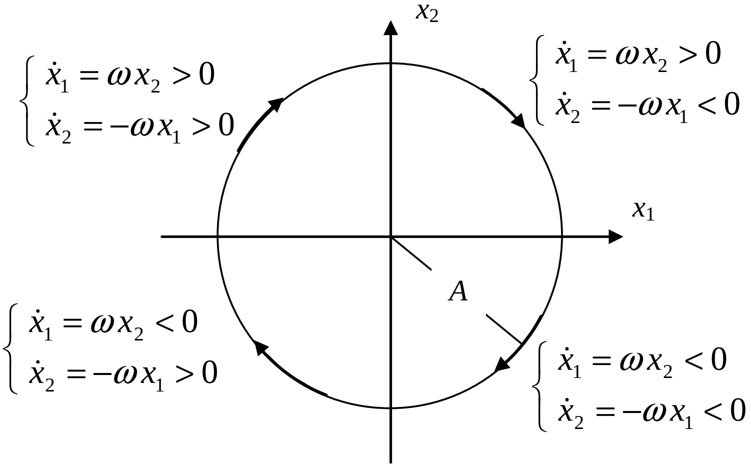

Figure-4

If the initial condition of (2) is changed, the radius $A = \sqrt{x_1^2(0) + \omega^2 x_2^2(0)}$ of the trajectory in Figure-4 will be different. To show the characteristics of a second order system, a portrait is often adopted to contain a set of trajectories, which are related to different initial conditions. For the sinusoidal oscillator in (2), its portrait is shown in Figure-5, a set of concentric circles.

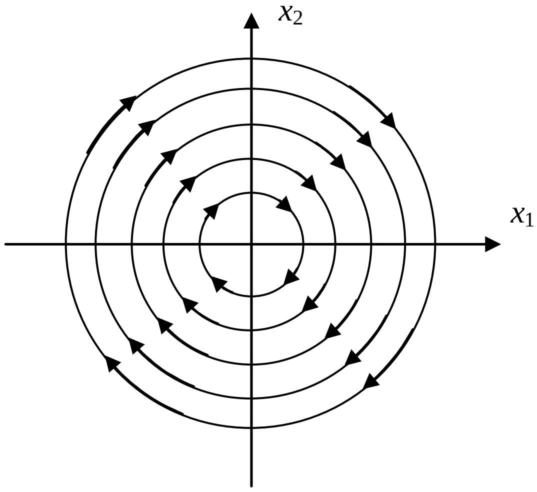

Figure-5

In reality, the ideal sinusoidal oscillator in Figure-3 is not usable due to the fact that energy dissipation inherently exists when using electronic components such as resistors, capacitors, and OPAmps. As a result, the trajectory of a physical sinusoidal oscillator does not follow a circle, but a converging spiral shown in Figure-6. The output signal $y(t)$ (or $x_1(t)$) in Figure-2 will converge fast and then vanish. To deal with the problem, compensation for energy dissipation is required to feedback. Many methods proposed for compensation have been employed in practice, such as the method of variable structure system, which will be introduced below.

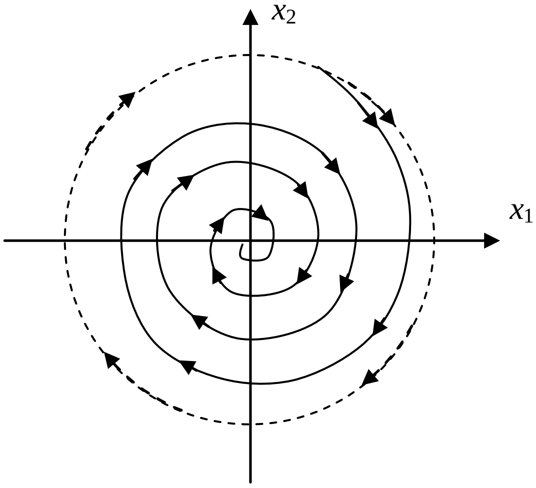

Figure-6

The use of variable structure system commonly requires at least two sub-systems. Here, two sub-systems will be employed to fulfill the sinusoidal oscillator, including the ideal sinusoidal oscillator (2), written again, as below:

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = -\omega x_1(t) \end{cases} \tag{3}$$

and a modified sub-system as following:

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = \omega x_1(t) \end{cases} \tag{4}$$

where the initial conditions of (3) and (4) are arbitrary. Later, it will be shown that no matter what the initial conditions are, the total system can be controlled by a switching technology to generate the sinusoidal output $x_1(t)$ with desired amplitude A and frequency ω. Further combine (3) and (4) into

$$\begin{cases} \dot{x}_1(t) = \omega x_2(t) \\ \dot{x}_2(t) = f\left(x_1(t), x_2(t)\right) \cdot \omega x_1(t) \end{cases} \tag{5}$$

where $f\left(x_1(t), x_2(t)\right) = -1$ for (3) and $f\left(x_1(t), x_2(t)\right) = 1$ for (4). Before explaining the switching function $f\left(x_1(t), x_2(t)\right)$, let's discuss the sub-system (4) in detail.

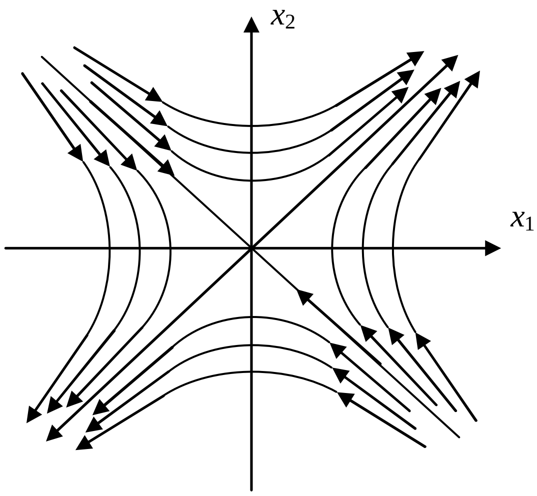

Figure-7

Since it is still true for (4) that $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)/\omega$, we can describe the system by the following differential equation

$$\ddot{y}(t) - \omega^2 y(t) = 0, \qquad \text{with } y(0) = y_0,\ \dot{y}(0) = \dot{y}_0 \tag{6}$$

which has two eigenvalues, ω and $-\omega$, and its solution is $y(t) = \alpha \cdot e^{\omega t} + \beta \cdot e^{-\omega t}$, where α and β depend on y_0 and $\dot{y}_0$. That implies

$$x_1(t) = \alpha \cdot e^{\omega t} + \beta \cdot e^{-\omega t} \quad \text{and} \quad x_2(t) = \alpha \cdot e^{\omega t} - \beta \cdot e^{-\omega t} \tag{7}$$

and thus, we have

$$x_1(t)+x_2(t)=2\alpha\cdot e^{\omega t} \quad \text{and} \quad x_1(t)-x_2(t)=2\beta\cdot e^{-\omega t} \tag{8}$$

It is obvious that

$$(x_1(t)+x_2(t))\cdot(x_1(t)-x_2(t))=\gamma \tag{9}$$

where $\gamma=4\alpha\beta$ depending on the initial conditions y_0 and $\dot{y}_0$. Note that the trajectory of (9) is a hyperbola in phase plane x_1-x_2. Its portrait is illustrated in Figure-7 where contains two asymptotes and a set of hyperbolas with different initial conditions.

The circuit of sub-system (4) can be directly obtained by modifying Figure-3 as the structure shown in Figure-8, where line-1 is disconnected and line-2 is connected.

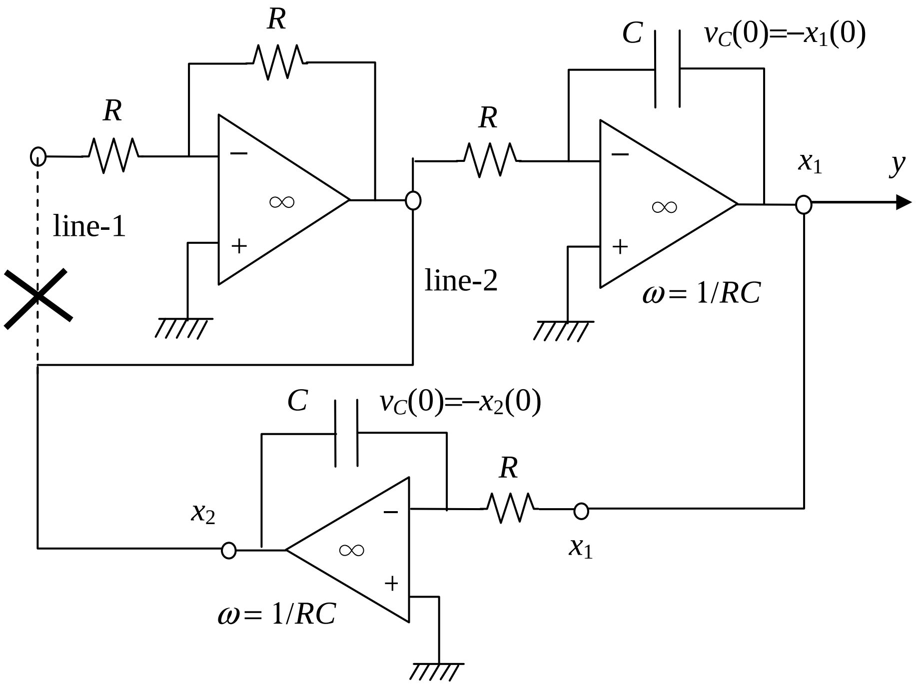

Figure-8

Now let's discuss the switching function $f(x_1(t),x_2(t))$. In this case, the switching is chosen as

$$f(x_1(t),x_2(t))=\begin{cases}-1 & (x_1(t),x_2(t))\in Q_1\cup Q_2\\ 1 & \text{elsewhere}\end{cases} \tag{10}$$

where Q_1 and Q_2 are shadowed in Figure-9. In addition to Q_1 and Q_2, Figure-9 also contains the portraits (dashed) of sub-systems (3) and (4) and the desired trajectory (solid) of ideal sinusoidal oscillator with magnitude *A*. Four possible trajectories of the total system (5) switched by (10) are demonstrated as bold-solid lines, numbered from 1 to 4. Obviously, all the possible trajectories move to the desired trajectory. Therefore, with the method of variable structure system to compensate the energy dissipation, the sinusoidal oscillator is successfully designed in Figure-10, which contains a switching box to achieve the function of (10).

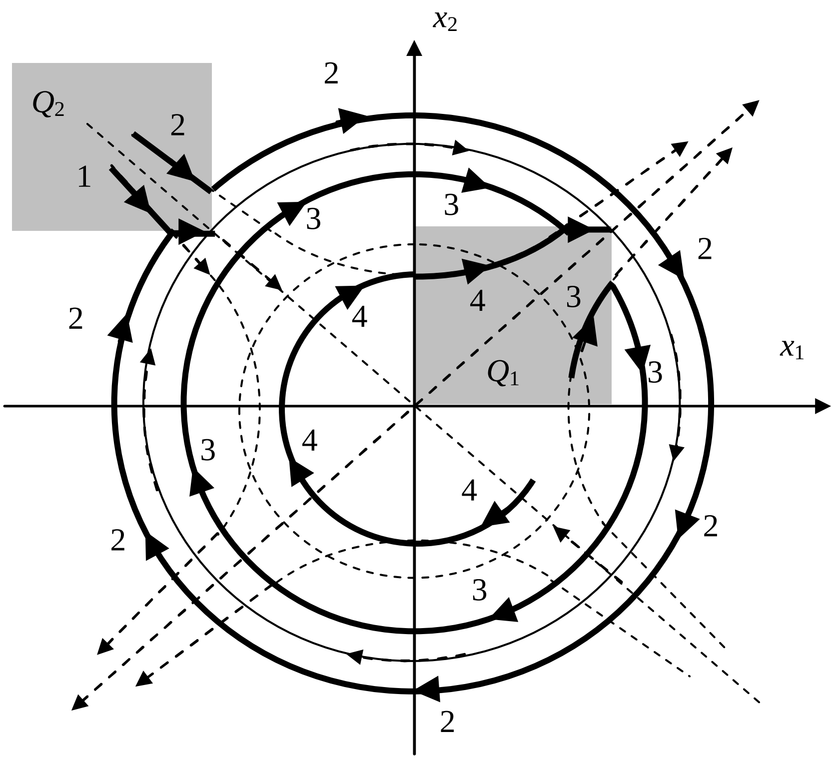

Figure-9

In the implementation of the sinusoidal oscillator (5), we have to face the problem caused by the precision limitation of physical components, which results in the following system

$$\begin{cases} \dot{x}_1(t) = \omega_1 x_2(t) \\ \dot{x}_2(t) = f\left(x_1(t), x_2(t)\right) \cdot \omega_2 x_1(t) \end{cases} \tag{11}$$

where $\omega_1 = \omega + \Delta\omega_1$ and $\omega_2 = \omega + \Delta\omega_2$, both with small deviation to the desired frequency ω. Then, setting $y(t) = x_1(t)$ and $f\left(x_1(t), x_2(t)\right) = -1$ yields

$$\ddot{y}(t) + \omega_1 \cdot \omega_2 y(t) = \ddot{y}(t) + \omega^2 \left(1 + \frac{\Delta\omega_1}{\omega}\right)\left(1 + \frac{\Delta\omega_2}{\omega}\right) y(t) = 0 \tag{12}$$

whose frequency is changed into $\omega\sqrt{\left(1 + \frac{\Delta\omega_1}{\omega}\right)\left(1 + \frac{\Delta\omega_2}{\omega}\right)}$ and deviated from ω about $\frac{1}{2\omega} \cdot \left|\Delta\omega_1 + \Delta\omega_2\right|$. Clearly, to achieve correct desired frequency, both $\Delta\omega_1$ and $\Delta\omega_2$ have to be small enough.

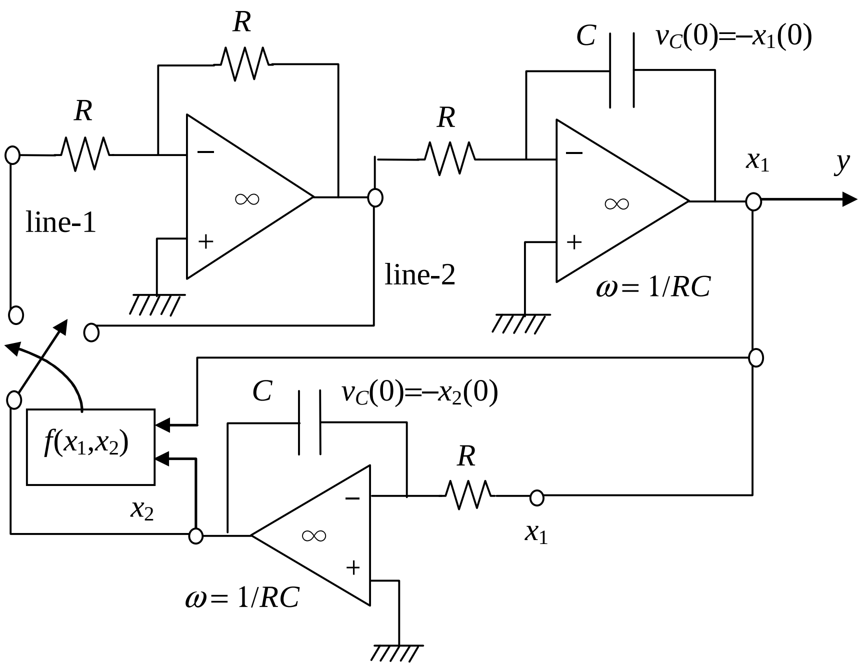

Figure-10

In addition to precision limitation of physical components, there is one other problem caused by the switching box which implements the switching condition $f\left(x_1(t), x_2(t)\right)$. Actually, it is impossible to setting the region Q_1 (or Q_2) with its corner exactly touching on the circular trajectory of the ideal sinusoidal oscillator with amplitude A. As a result, an abrupt variation may happen around the corners of Q_1 and Q_2. The output at the terminal $x_1(t)$ may be shown as the waveform in Figure-11, with sharp tooth around the amplitude of $0.707A$ and $-0.707A$. It is called the Gibbs phenomenon. Fortunately, we can use a low-pass filter connected to the terminal $x_1(t)$ to make the sharp tooth smoother.

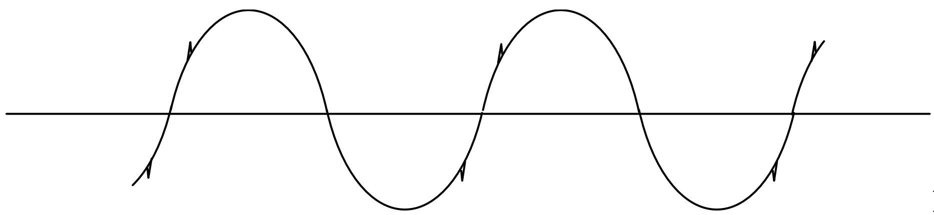

Figure-11

P.1 Simulate the sinusoidal oscillator introduced in this topic by Simulink in Matlab. You have to verify your results with different initial conditions.