16. DC Motor – Actuator in Servo-Control

The most common device used as an actuator in mechanical control is the DC motor. For example, the control of an inverted pendulum requires a DC motor to drive the cart through the belt, as shown in Figure-1.

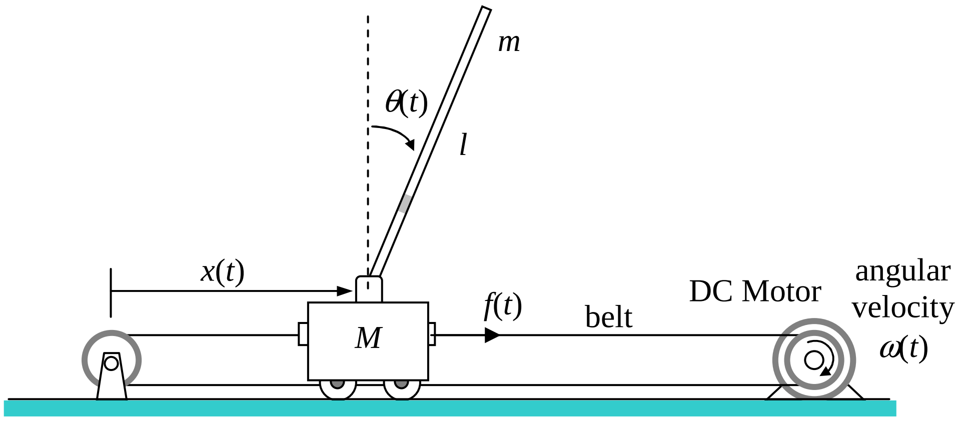

Figure-1

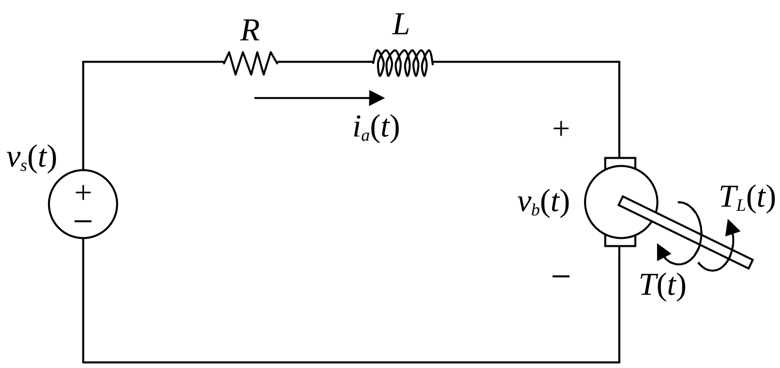

Figure-2

The system structure of a DC motor is depicted in Figure-2, including the armature resistance *R* and winding leakage inductance *L*. According to the Kirchhoff's voltage law, the electrical equation of the DC motor is described as

$$Ri_a(t) + L\frac{di_a(t)}{dt} + v_b(t) = v_s(t) \tag{1}$$

where $i_a(t)$ is the armature current, $v_b(t)$ is the back emf voltage and $v_s(t)$ is the voltage source. The back emf voltage $v_b(t)$ is proportional to the angular velocity $\omega(t)$ of the rotor in the motor, expressed as

$$v_b(t) = k_b \omega(t) \tag{2}$$

where k_b is the back emf constant. In addition, the motor generates torque T to operate payloads, which is proportional to the armature current and given as below:

$$T(t) = k_T i_a(t) \tag{3}$$

where k_T is the torque constant. It is known that k_b is equal to k_T; usually. let $k = k_b = k_T$ The mechanical equation is then described as

$$J \frac{d\omega(t)}{dt}(t) + B\omega(t) = T(t) - T_L(t) \tag{4}$$

where J is the rotor moment of inertia, B is the frictional coefficient and $T_L(t)$ is the external torque from payloads.

Based on the above equations, the dynamic equation of the motor can be expressed as

$$Ri_a(t) + L\frac{di_a(t)}{dt} + k\omega(t) = v_s(t) \tag{5}$$

$$J \frac{d\omega(t)}{dt}(t) + B\omega(t) - ki_a(t) = -T_L(t) \tag{6}$$

Note that the electrical time constant L/R is often neglected since it is usually at least an order of magnitude smaller than the mechanical time constant J/B. In other words, the variation $\frac{di_a(t)}{dt}$ is omitted in (5) and then

$$i_a(t) = \frac{1}{R} v_s(t) - \frac{k}{R}\omega(t) \tag{7}$$

Substituting it into (6), we have

$$\frac{d\omega(t)}{dt}(t) + \left(\frac{B}{J} + \frac{1}{JR}k^2\right)\omega(t) = -\frac{1}{J}T_L(t) + \frac{1}{JR}kv_s(t) \tag{8}$$

Clearly, the motor encounters two external sources. One is the input voltage $v_s(t)$ to drive the motor and the other is the torque $T_L(t)$ reacted from the payload.

Now, let's use the DC motor as the actuator to control the inverted pendulum. It is known that the torque $T_L(t)$ from the payload is proportion to the force $f(t)$ to drive the cart, i.e.,

$$T_L(t) = \alpha f(t) \tag{9}$$

where α is a constant. Besides, the actuator and the cart interacts under the constraint of the belt and thus the following condition must be satisfied:

$$\dot{x}(t) = \beta\omega(t) \tag{10}$$

Hence, (8) can be rewritten as

$$\frac{J}{\alpha\beta}\ddot{x}(t) + \left(\frac{B}{\alpha\beta} + \frac{k^2}{\alpha\beta R}\right)\dot{x}(t) = -f(t) + \frac{k_T}{\alpha R}v_s(t) \tag{11}$$

which will be combined into the dynamic equations of the inverted pendulum derived before, expressed as below:

$$(M+m)\ddot{x}(t) + \frac{ml}{2}cos\theta(t)\ddot{\theta}(t) - \frac{ml}{2}sin\theta(t)\dot{\theta}^2(t) = f(t) \tag{12}$$

$$\frac{ml}{2}cos\theta(t)\ddot{x}(t) + \frac{ml^2}{3}\ddot{\theta}(t) - mg\frac{l}{2}sin\theta(t) = 0 \tag{13}$$

From (11) and (12), we have

$$\left(M + m + \frac{J}{\alpha\beta}\right)\ddot{x}(t) + \frac{ml}{2}cos\theta(t)\ddot{\theta}(t) - \frac{ml}{2}sin\theta(t)\dot{\theta}^2(t) + \left(\frac{B}{\alpha\beta} + \frac{k^2}{\alpha\beta R}\right)\dot{x}(t) = \frac{k}{\alpha R}v_s(t) \tag{14}$$

If the following motor parameters are used: α=0.04 m, β=0.02 m/rad, R=18.6 Ω, L=6.6 mH, J=8×10^{-7} kg-m-s2, k=0.1738 kg-m/A and B is negligible, then

$$\frac{J}{\alpha\beta} = 10^{-3},\ \frac{B}{\alpha\beta} = 0,\ \frac{k^2}{\alpha\beta R} = 2.030,\ \frac{k}{\alpha R} = 0.2336$$

Clearly, (14) can be approximated as

$$(M+m)\ddot{x}(t) + \frac{ml}{2}cos\theta(t)\ddot{\theta}(t) - \frac{ml}{2}sin\theta(t)\dot{\theta}^2(t) + \frac{k^2}{\alpha\beta R}\dot{x}(t) = \frac{k}{\alpha R}v_s(t) \tag{15}$$

which is different to (12) by adding a small term $\frac{k^2}{\alpha\beta R}\dot{x}(t)$ and changing the input $f(t)$ into $\frac{k}{\alpha R}v_s(t)$. In case that the term $\frac{k^2}{\alpha\beta R}\dot{x}(t)$ can be further omitted, the inverted pendulum driven by the motor is expressed as below:

$$(M+m)\ddot{x}(t)+\frac{ml}{2}cos\theta(t)\ddot{\theta}(t)-\frac{ml}{2}sin\theta(t)\dot{\theta}^2(t)=\frac{k}{\alpha R}v_s(t) \quad (16)$$

$$\frac{ml}{2}cos\theta(t)\ddot{x}(t)+\frac{ml^2}{3}\ddot{\theta}(t)-mg\frac{l}{2}sin\theta(t)=0 \quad (17)$$

which can be linearized under the assumption that $\theta(t)\approx 0$ and described as

$$(M+m)\ddot{x}(t)+\frac{ml}{2}\ddot{\theta}(t)=\frac{k}{\alpha R}v_s(t) \quad (18)$$

$$\frac{ml}{2}\ddot{x}(t)+\frac{ml^2}{3}\ddot{\theta}(t)-mg\frac{l}{2}\theta(t)=0 \quad (19)$$

Therefore, according to the pole-placement method, the state-feedback control can be found as

$$\frac{k}{\alpha R}v_s(t)=k_1x(t)+k_2\theta(t)+k_3\dot{x}(t)+k_4\theta(t) \quad (20)$$

and thus, the control input voltage is set to be

$$v_s(t)=\frac{\alpha R}{k}(k_1x(t)+k_2\theta(t)+k_3\dot{x}(t)+k_4\theta(t)) \quad (21)$$

which could control the inverted pendelum successfully.

P.1 Consider an inverted pendulum with parameters: M=1kg, m=0.8kg, l=2m, which is driven by a DC motor with parameter α=0.04 m, β=0.02 m/rad, R=18.6 Ω, L=6.6 mH, $J=8\times10^{-7}$ kg-m-s2, k= 0.1738 kg-m/A and B is negligible. Please design a state-feedback control to drive the inverted pendulum to $x(t)$=0 and $\theta(t)$=0 and verify your controller by SIMUINK for (13) and (15) with initial conditions $x(t)$=0.5, $\dot{x}(0)=0$, $\theta(0)$=0.2 and $\dot{\theta}(0)=0$.