13. State-feedback Control of LTI Systems

From this section, we will focus on the state-feedback control applied to an LTI system in state-space description, which is expressed as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{2}$$

where $\boldsymbol{u}(t)=[u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T \in \Re^m$ is the input, $\boldsymbol{y}(t)=[y_1(t)\ y_2(t)\ \cdots\ y_p(t)]^T \in \Re^p$ is the output and $\boldsymbol{x}(t)=[x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \Re^n$ is the system state with initial condition $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$ at the initial time $t=t_0$.

First, let's consider the stability problem. It is well-known that if the system satisfies the following condition

$$rank\begin{bmatrix}\boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{bmatrix} = n \tag{3}$$

then the system is controllable, i.e., with the use of state feedback

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}(t) \tag{4}$$

the system state $\boldsymbol{x}(t)$ will approach the origin $\boldsymbol{0}$ from any initial state $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$.

Next, let's show how to determine the matrix $\boldsymbol{K}$. Commonly, the so-called pole-placement method is adopted. With the use of (4), the state equation is changed into

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}(t) \tag{5}$$

which implies that the system can be stabilized when the eigenvalues of $\boldsymbol{A}-\boldsymbol{BK}$ are located in the left-half complex plane, i.e., each eigenvalue possesses a negative real part. To apply the pole-placement method, first we have to choose n desired eigenvalues, such as λ_i, $i=1,2,\ldots,n$. Then, determine $\boldsymbol{K}$ from the following characteristic polynomial

$$\left|\lambda\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right| = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \tag{6}$$

Note that the matrix $\boldsymbol{K}$ is not unique unless $m=1$. In other words, $\boldsymbol{K}$ is unique only for single input systems, not for multiple input systems. By adding other limitation on ***K***, the pole-placement method can be used to uniquely determine the matrix ***K***. Now, let's consider a system with four state variables and two inputs, described as below:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{7}$$

with initial condition $\boldsymbol{x}(0) = \begin{bmatrix} 5 & 4 & -2 & 1 \end{bmatrix}^T$. First, let's check the system stability without any input, i.e., check the eigenvalues of ***A***. By the use of MATLAB, we have

```
==========================================

>>% Create A and check eigenvalues of A
>> A=[1 0 -1 0; 0 -1 0 1; -1 0 1 1; 0 1 -1 0];
>> eig(A)

ans =

   1.6903
   0.4074 + 0.4766i
   0.4074 - 0.4766i
  -1.5051

==========================================
```

Clearly, the eigenvalues are not all located in the left-half complex plane, which means the system is unstable without input. To stabilize the system, before the use of state feedback control (4), we have to check whether the system is controllable or not by the condition (3). From MATLAB, we have

```
==========================================

>>% Create B and check rank[B AB A^2B A^3B]
>> B=[0 0; 0 0; 1 0; 0 1];
>> rank([B A*B A^2*B A^3*B])

ans =

   4

==========================================
```

Evidently, the system satisfies the condition (3) and thus, it is controllable. Then, based on the pole-placement method, we assign four eigenvalues −1, −2±j2, −4 for the matrix ***A*–*BK*** whose character polynomial is

$$\left|\lambda \boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{BK})\right| = (\lambda+1)(\lambda+2+j2)(\lambda+2-j2)(\lambda+4) \tag{8}$$

and solve ***K*** by MATLAB as below:

```
==========================================
>>% Solve K by pole-placement method
>> p=[-1 -2+2j -2-2j -4];
>> K=place(A, B, p)

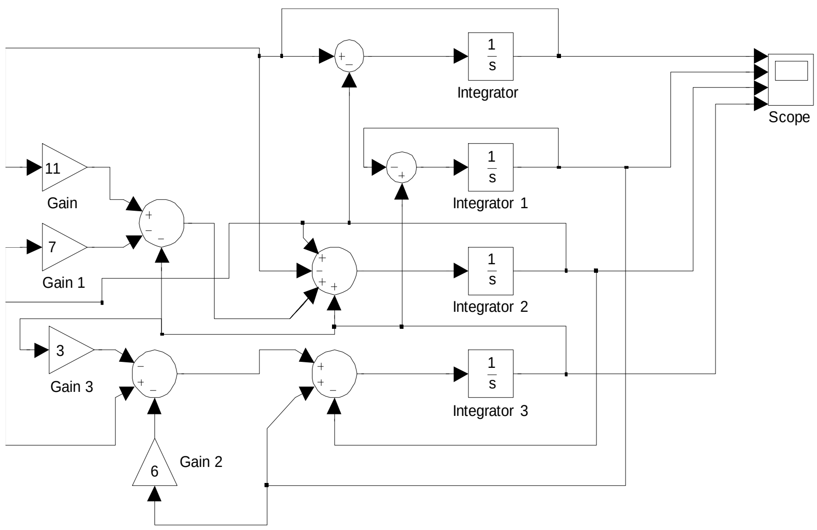
K =

   -11.0000         0    7.0000    1.0000
          0    6.0000   -1.0000    3.0000
==========================================
```

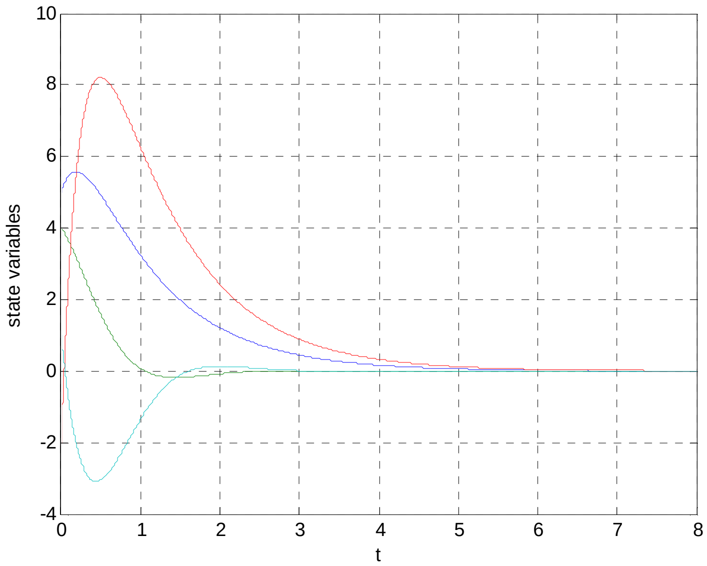
The control inputs are expressed as below:

$$\begin{aligned} u_1(t) &= 11x_1(t) - 7x_3(t) - x_4(t) \\ u_2(t) &= -6x_2(t) + x_3(t) - 3x_4(t) \end{aligned} \tag{9}$$

By the use of SIMULINK, we have the block diagram as below:



The numerical result of the four state variables can be obtaine from the Scope block and shown as below:



Clearly, the system is stabilized since all the state variables are driven to approach the origin under state-feedback control.

Next, let's consider the so-called regulation problem: Drive the output $\boldsymbol{y}(t)$ to the fixed destination $\boldsymbol{y}_d$, i.e., $\boldsymbol{y}(t)=\boldsymbol{y}_d$. To deal with such problem, define the error vector as

$$\boldsymbol{e}(t)=\boldsymbol{y}(t)-\boldsymbol{y}_d=\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d \tag{10}$$

and genetate a new state by the use of integrator as below:

$$\boldsymbol{z}(t)=\int_0^t \boldsymbol{e}(\tau)d\tau=\int_0^t(\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d)d\tau \tag{11}$$

which implies

$$\dot{\boldsymbol{z}}(t)=\boldsymbol{e}(t)=\boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{y}_d\,,\qquad \boldsymbol{z}(t_0)=\boldsymbol{0} \tag{12}$$

Combined with (1), it leads to the following augmented system

$$\underbrace{\begin{bmatrix}\dot{\boldsymbol{x}}(t)\\ \dot{\boldsymbol{z}}(t)\end{bmatrix}}_{\dot{\hat{\boldsymbol{x}}}(t)}=\underbrace{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{C} & \boldsymbol{0}\end{bmatrix}}_{\hat{\boldsymbol{A}}}\cdot\underbrace{\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{z}(t)\end{bmatrix}}_{\hat{\boldsymbol{x}}(t)}+\underbrace{\begin{bmatrix}\boldsymbol{B}\\ \boldsymbol{0}\end{bmatrix}}_{\hat{\boldsymbol{B}}}\boldsymbol{u}(t),\qquad \underbrace{\begin{bmatrix}\boldsymbol{x}(t_0)\\ \boldsymbol{z}(t_0)\end{bmatrix}}_{\hat{\boldsymbol{x}}(t_0)}=\begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{0}\end{bmatrix} \tag{13}$$

It has been proved that if $m \geq p$ and (3) is satisfied then

$$rank\begin{bmatrix}\hat{\boldsymbol{B}} & \hat{\boldsymbol{A}}\hat{\boldsymbol{B}} & \hat{\boldsymbol{A}}^2\hat{\boldsymbol{B}} & \cdots & \hat{\boldsymbol{A}}^{n+p-1}\hat{\boldsymbol{B}}\end{bmatrix}=n+p \tag{14}$$

i.e., the augmented system (13) is also controllable. Hence, with the use of state feedback

$$\boldsymbol{u} = -\hat{\boldsymbol{K}}\hat{\boldsymbol{x}}(t) \tag{15}$$

the system state $\hat{\boldsymbol{x}}(t)$ will approach the origin $\boldsymbol{0}$ from the initial state $\hat{\boldsymbol{x}}(t_0) = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix}$.

Based on the pole- placement method, the augmented state equation is changed into

$$\dot{\hat{\boldsymbol{x}}}(t) = \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\hat{\boldsymbol{x}}(t) \tag{16}$$

where the eigenvalues of $\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}$ are assigned as $\hat{\lambda}_i$, i=1,2,…,n+p, all located in the left-half complex plane. Then, determine $\hat{\boldsymbol{K}}$ by the pole-placement method from the following characteristic polynomial

$$\left|\lambda \boldsymbol{I} - \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\right| = \left(\lambda - \hat{\lambda}_1\right)\left(\lambda - \hat{\lambda}_2\right)\cdots\left(\lambda - \hat{\lambda}_{n+p}\right) \tag{17}$$

It will be demonstrated by the following example:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}}_{\boldsymbol{u}(t)} \tag{18}$$

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_{\boldsymbol{y}(t)} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{C}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} \tag{19}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = [5 \quad 4 \quad -2 \quad 1]^T$. Let the destination be $\boldsymbol{y}_d = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then we can choose the error vector as

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{y}_d = \begin{bmatrix} y_1(t) - 1 \\ y_2(t) + 1 \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t) + x_4(t) - 1 \\ -x_1(t) + x_4(t) + 1 \end{bmatrix} \tag{20}$$

Further genetate a new state as (11) to construct the augmented system as below:

$$\underbrace{\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{z}}(t) \end{bmatrix}}_{\dot{\hat{\boldsymbol{x}}}(t)} = \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}}_{\hat{\boldsymbol{A}}} \cdot \underbrace{\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{z}(t) \end{bmatrix}}_{\hat{\boldsymbol{x}}(t)} + \underbrace{\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}}_{\hat{\boldsymbol{B}}} \boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{y}_d \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{z}(0) \end{bmatrix}}_{\hat{\boldsymbol{x}}(0)} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix} \quad (21)$$

First, let's check the condition (14) to make sure whether the system is controllable or not. From MATLAB, we have

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  >>% Create Ah and Bh; Check rank[Bh AhBh … Ah^5Bh ]
  >> Ah=[1 0 -1 0 0 0; 0 -1 0 1 0 0; -1 0 1 1 0 0; 0 1 -1 0 0 0;
           1 -1 0 1 0 0; -1 0 0 1 0 0];
  >> Bh=[0 0; 0 0; 1 0; 0 1; 0 0; 0 0];
  >> rank([Bh Ah*Bh Ah^2*Bh Ah^3*Bh Ah^4*Bh Ah^5*Bh])

  ans =

       6
============================================
```

As expected, the augmented system is also controllable. Then, based on the pole-placement method, we assign six eigenvalues –1, –2±j2, –3±j3, –4 for the matrix $\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}$ whose character polynomial is

$$\left|\lambda \boldsymbol{I} - \left(\hat{\boldsymbol{A}} - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\right| = (\lambda+1)(\lambda+2+j2)(\lambda+2-j2) \\ (\lambda+3+j3)(\lambda+3-j3)(\lambda+4) \quad (22)$$

and solve $\hat{\boldsymbol{K}}$ by MATLAB as below:

```
============================================
  >>% Solve K by pole-placement method
  >> p=[-1 -2+2j -2-2j -3+3j -3-3j -4];
  >> Kh=place(Ah, Bh, p)

  Kh =

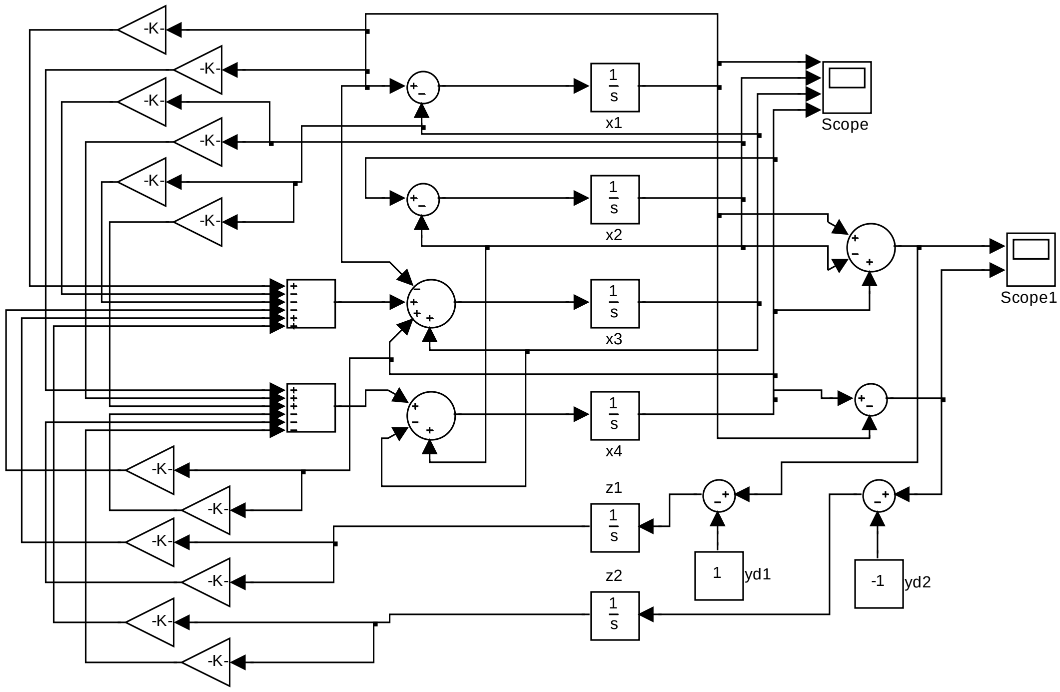
      -35.5688    36.7645    10.0393     0.7304   -34.1787    -1.6543
      -1.3863    -24.6179    -1.1292     5.9607    24.1804    18.0229
============================================
```

The control inputs are expressed as below:

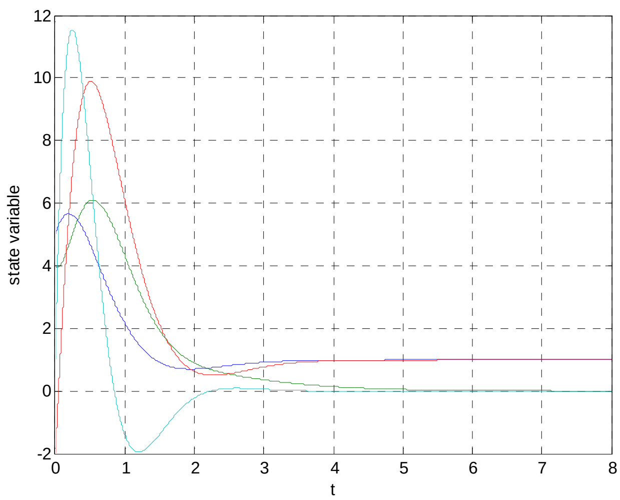
$$u_1(t) = 35.57x_1(t) - 36.76x_2(t) - 10.04x_3(t) - 0.73x_4(t) + 34.18z_1(t) + 1.65z_2(t) \quad (23)$$

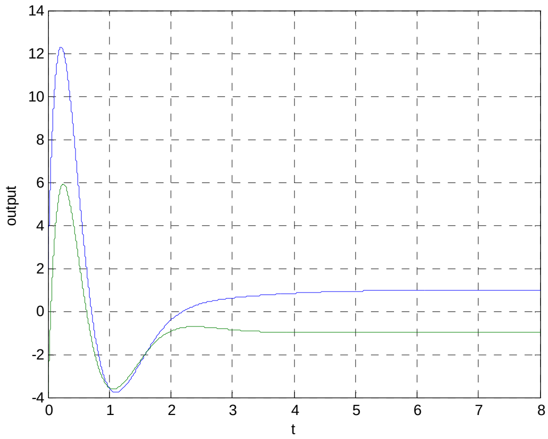
$$u_2(t) = 1.39x_1(t) + 24.62x_2(t) + 1.13x_3(t) - 5.96x_4(t) - 24.18z_1(t) - 18.02z_2(t) \quad (24)$$

By the use of SIMULINK, we have the block diagram as below:



The numerical result of the four state variables can be obtaine from the Scope block and shown as below:





Clearly, the system outputs $y_1(t)$ and $y_2(t)$ are successfully regulated to $y_{1d}=1$ and $y_{2d}=-1$.

P.1 Consider the following system:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}}_{\boldsymbol{A}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\boldsymbol{b}} \cdot u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}}_{\boldsymbol{C}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}}_{\boldsymbol{x}(t)}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} 1 & 0 & -2 & 1 \end{bmatrix}^T$. Please solve the regulation problem by state-feedback control to drive the output to $y_d = 1$.