9. Second-Order Filter Design

In engineering, a filter is often used to acquire or reject signals of special frequency domain. There are three fundamental types of filters, called low-pass, high-pass and band-pass filters. Let's employ a second order LTI system for filter design, which is expressed by the following input-output equation

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0 y(t)=b_2\ddot{u}(t)+b_1\dot{u}(t)+b_0 u(t) \tag{1}$$

where $a_1>0$ and $a_0>0$. Taking The Laplace transform, we have the transfer function as

$$H(s)=\frac{Y(s)}{U(s)}=\frac{b_2 s^2+b_1 s+b_0}{s^2+a_1 s+a_0} \tag{2}$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of the input $u(t)$ and output $y(t)$.

Assign s_1 and s_2, located on the left-half complex plane due to $a_1>0$ and $a_0>0$, as the roots of the characteristic equation, then

$$s^2+a_1 s+a_0=(s-s_1)(s-s_2)=0 \tag{3}$$

Since both s_1 and s_2 possess negative real part, the system (1) is stable and the effect caused by the initial conditions will vanish as $t\to\infty$. Therefore, it is not necessary to consider the initial conditions in the filter design.

Commonly, there are three fundamental filters concerning the second-order LTI system (1), named low-pass, high-pass and band-pass filters. The low-pass filter is of the form

$$\ddot{y}_{LP}(t)+a_1\dot{y}_{LP}(t)+a_0 y_{LP}(t)=b_0 u(t) \tag{4}$$

where $b_0=k_0 a_0$ and $k_0>0$ is the gain. The resulted transfer function is

$$H_{LP}(s)=\frac{Y_{LP}(s)}{U(s)}=k_0\frac{a_0}{s^2+a_1 s+a_0} \tag{5}$$

Let $s=j\omega$, then

$$H_{LP}(j\omega)=\frac{Y_{LP}(j\omega)}{U(j\omega)}=k_0\frac{a_0}{(j\omega)^2+a_1(j\omega)+a_0} \tag{6}$$

which implies

$$H_{LP}(j\omega)\approx k_0\angle 0^\circ \qquad \text{as } \omega\to 0 \tag{7}$$

$$H_{LP}(j\omega)\approx 0\angle -180^\circ \qquad \text{as } \omega\to\infty \tag{8}$$

Clearly, if the input is a sinusoidal signal given as

$$u(t) = A\cos(\omega t + \theta) \tag{9}$$

with magnitude A and phase θ, then its output is

$$y_{LP}(t) = A\left|H_{LP}(j\omega)\right|\cos(\omega t + \theta + \angle H_{LP}(j\omega)) \tag{10}$$

with magnitude changed by the gain $\left|H_{LP}(j\omega)\right|$ and phase shifted by $\angle H_{LP}(j\omega)$. Moreover, from (7) and (8) we have

$$y_{LP}(t) \approx Ak_0\cos(\omega t + \theta) \qquad \text{as } \omega \to 0 \tag{11}$$

$$y_{LP}(t) \approx 0 \cdot \cos(\omega t + \theta - 180^\circ) \qquad \text{as } \omega \to \infty \tag{12}$$

i.e., the low-pass filter rejects inputs of high frequency and allows low-frequency inputs to pass through.

In a certain cases, if we need a system to receive high-frequency inputs and reject those of low frequency, then the following high-pass filter can reach the goal:

$$\ddot{y}_{HP}(t) + a_1\dot{y}_{HP}(t) + a_0 y_{HP}(t) = b_2\ddot{u}(t) \tag{13}$$

where $b_2=k_2>0$ is the gain. The transfer function is

$$H_{HP}(s) = \frac{Y_{HP}(s)}{U(s)} = k_2 \frac{s^2}{s^2 + a_1 s + a_0} \tag{14}$$

Hence,

$$H_{HP}(j\omega) = \frac{Y_{HP}(j\omega)}{U(j\omega)} = k_2 \frac{(j\omega)^2}{(j\omega)^2 + a_1(j\omega) + a_0} \tag{15}$$

which implies

$$H_{HP}(j\omega) \approx 0\angle 180^\circ \qquad \text{as } \omega \to 0 \tag{16}$$

$$H_{HP}(j\omega) \approx k_2\angle 0^\circ \qquad \text{as } \omega \to \infty \tag{17}$$

The sinusoidal input (9) will generate the following output

$$y_{HP}(t) = A\left|H_{HP}(j\omega)\right|\cos(\omega t + \theta + \angle H_{HP}(j\omega)) \tag{18}$$

Similarly, we have

$$y_{HP}(t) \approx 0 \cdot \cos(\omega t + \theta + 180^\circ) \qquad \text{as } \omega \to 0 \tag{19}$$

$$y_{HP}(t) \approx Ak_2\cos(\omega t + \theta) \qquad \text{as } \omega \to \infty \tag{20}$$

i.e., the high-pass filter (13) rejects low-frequency inputs, but allows high-frequency inputs to pass through.

If both the low-frequency and high-frequency inputs are unwanted, then a band-pass filter is required to achieve the objective, constructed as the following form:

$$\ddot{y}_{BP}(t)+a_1\dot{y}_{BP}(t)+a_0y_{BP}(t)=b_1\dot{u}(t) \tag{21}$$

where $b_1=k_1a_1$ and k_1>0 is the gain. The transfer function is

$$H_{BP}(s)=\frac{Y_{BP}(s)}{U(s)}=k_1\frac{a_1s}{s^2+a_1s+a_0} \tag{22}$$

Therefore,

$$H_{BP}(j\omega)=\frac{Y_{BP}(j\omega)}{U(j\omega)}=k_1\frac{a_1(j\omega)}{(j\omega)^2+a_1(j\omega)+a_0} \tag{23}$$

which implies

$$H_{BP}(j\omega)\approx 0\angle 90^\circ \qquad \text{as } \omega\to 0 \tag{24}$$

$$H_{BP}(j\omega)=k_1\angle 0^\circ \qquad \text{at } \omega=\sqrt{a_0} \tag{25}$$

$$H_{BP}(j\omega)\approx 0\angle -90^\circ \qquad \text{as } \omega\to\infty \tag{26}$$

With the use of sinusoidal input (9), the output is

$$y_{BP}(t)=A|H_{BP}(j\omega)|cos(\omega t+\theta+\angle H_{BP}(j\omega)) \tag{27}$$

which leads to

$$y_{BP}(t)\approx 0\cdot cos(\omega t+\theta+90^\circ) \qquad \text{as } \omega\to 0 \tag{28}$$

$$y_{BP}(t)\approx Ak_1cos(\omega t+\theta) \qquad \text{at } \omega=\sqrt{a_0} \tag{29}$$

$$y_{BP}(t)\approx 0\cdot cos(\omega t+\theta-90^\circ) \qquad \text{as } \omega\to\infty \tag{30}$$

Clearly, the band-pass filter rejects both the low-frequency and high-frequency inputs.

To implement the above 2nd order filters, we adopt a basic system with the state equation described as below:

$$\dot{x}_1(t)=x_2(t) \tag{31}$$

$$\dot{x}_2(t)=-a_0x_1(t)-a_1x_2(t)+u(t) \tag{32}$$

If the output is $y_{LP}(t)=b_0x_1(t)$ where $b_0=k_0a_0$, then the input-output equation can be obtained as

$$\ddot{y}_{LP}(t)+a_1\dot{y}_{LP}(t)+a_0y_{LP}(t)=b_0u(t) \tag{33}$$

Clearly, it is a low-pass filter with the same expression as (4).

If the output is chosen as $y_{HP}(t)=b_2\dot{x}_2(t)=k_2(u(t)-a_0x_1(t)-a_1x_2(t))$, where $b_2=k_2$, then the input-output equation can be obtained as

$$\ddot{y}_{HP}(t)+a_1\dot{y}_{HP}(t)+a_0y_{HP}(t)=b_2\ddot{u}(t) \tag{34}$$

Clearly, it is a high-pass filter with the same expression as (13).

If the output is chosen as $y_{BP}(t)=b_1x_2(t)$ where $b_1=k_1a_1$, then the input-output equation can be obtained as

$$\ddot{y}_{BP}(t)+a_1\dot{y}_{BP}(t)+a_0y_{BP}(t)=b_1\dot{u}(t) \tag{35}$$

Clearly, it is a band-pass filter with the same expression as (21). Figure-1 shows the basic system described by (31) and (32) and the outputs $y_{LP}(t)$, $y_{HP}(t)$ and $y_{BP}(t)$ for low-pass, high-pass and band-pass filters.

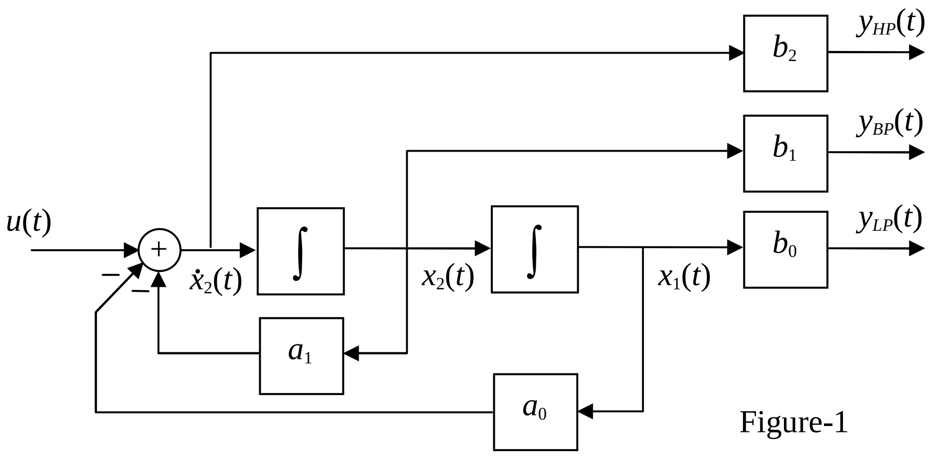


Figure-1

For numerical demonstration, we choose s_1=–10 and s_2=–100 for the filters discussed below. The resulted transfer function is

$$H(s)=\frac{b_2s^2+b_1s+b_0}{s^2+110s+1000}=\frac{b_2s^2+b_1s+b_0}{(s+10)(s+100)} \tag{36}$$

Let the coefficients be b_2=0, b_1=0 and b_0=1000, then (36) is a low-pass filter of the following form

$$H(j\omega)=\frac{1000}{(j\omega+10)(j\omega+100)}=\frac{1}{\left(1+j\dfrac{\omega}{10}\right)\left(1+j\dfrac{\omega}{100}\right)} \tag{37}$$

Clearly, the magnitude $|H(j\omega)|$ decreases monotonically as ω increases. Besides, the maximum magnitude is $|H(j\omega)|=1$ at ω=0 and $|H(j\omega)|\to 0$, as $\omega\to\infty$.

If the coefficients are b_2=1, b_1=0 and b_0=0, then (36) is a high-pass filter of the following form

$$H(j\omega)=\frac{(j\omega)^2}{(j\omega+10)(j\omega+100)}=\frac{1}{\left(1-j\dfrac{10}{\omega}\right)\left(1-j\dfrac{100}{\omega}\right)} \tag{38}$$

where the maximum magnitude $|H(j\omega)|=1$ exists at $\omega\to\infty$. Besides, the magnitude $|H(j\omega)|$ increases monotonically as ω increases and $|H(j\omega)|\to 0$, as $\omega\to 0$.

If the coefficients are b_2=0, b_1=110 and b_0=0, then (36) is a band-pass filter of the following form

$$\begin{aligned} H(j\omega)&=\frac{110j\omega}{(j\omega)^2+110j\omega+1000} \\ &=\frac{1}{1+j\dfrac{\sqrt{1000}}{110}\left(\dfrac{\omega}{\sqrt{1000}}-\dfrac{\sqrt{1000}}{\omega}\right)} \end{aligned} \tag{39}$$

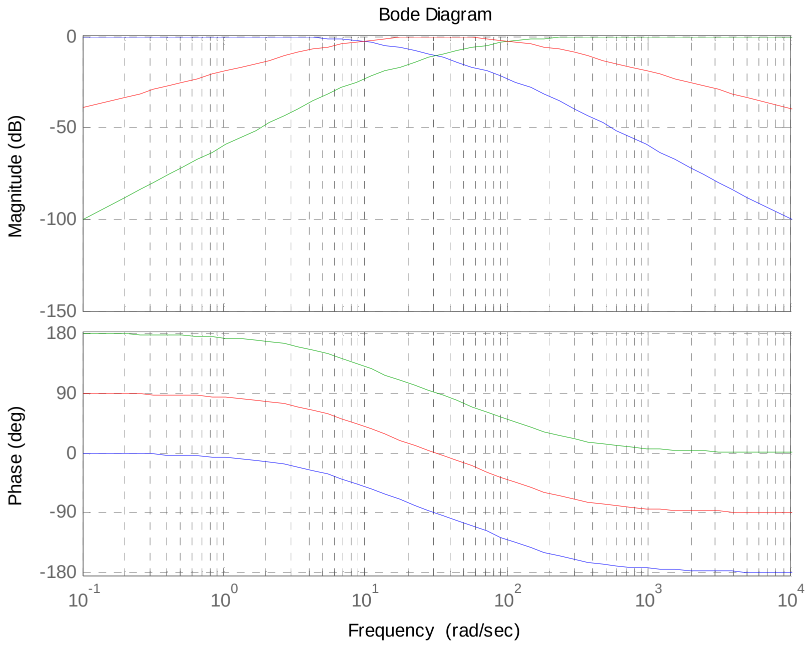
where the maximum magnitude $|H(j\omega)|=1$ at $\omega=\sqrt{1000}$. For $\omega<\sqrt{1000}$, the magnitude $|H(j\omega)|$ increases monotonically as ω increases. For $\omega>\sqrt{1000}$, the magnitude $|H(j\omega)|$ decreases monotonically as ω increases.

The Bode plots of these three filters can be obtained by the use of MATLAB, which are shown as below:

==

```
>>% Create transfer function
>>numlp=[0 0 1000]; numhp=[1 0 0]; numbp=[0 110 0]; den=[1 110 1000];
>>lpf=tf(numlp,den)

Transfer function:
        1000
--------------------------
s^2 + 110 s + 1000

>>hpf=tf(numhp,den)

Transfer function:
           s^2
------------------------
s^2 + 110 s + 1000

>>bpf=tf(numbp,den)

Transfer function:
```

```
        110s
------------------------
s^2 + 110 s + 1000

>>bode(lpf,hpf,bpf);grid
```



==

To show the property of these filters, let the input be a signal composed of three frequencies, expressed as

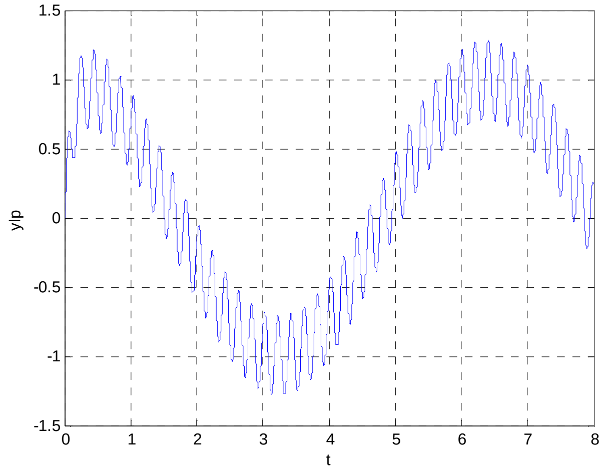
$$u(t) = cos(t) + cos\left(\sqrt{1000}t\right) + cos(1000t) \tag{40}$$

where ω=1 rad/sec is the low-frequency signal, ω=1000 rad/sec is the high-frequency signal and $\omega=\sqrt{1000}$ is of the middle frequency. The outputs $y_{LP}(t)$, $y_{HP}(t)$ and $y_{BP}(t)$ are obtained by the use of MABLAB.

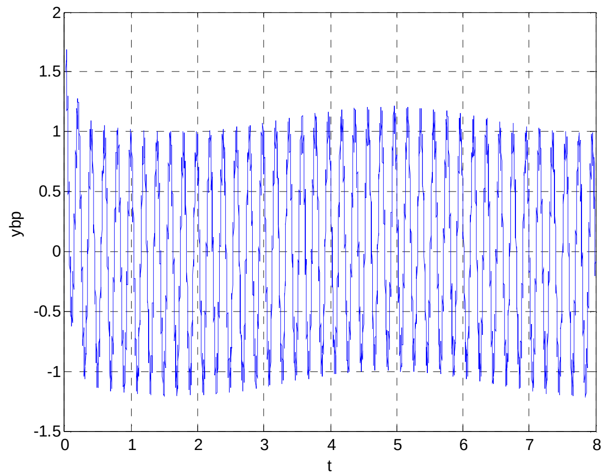
==

```
Create m-file: second.m
  function dx=second(t,x)
  dx=zeros(2,1);
  dx(1)=x(2);
  dx(2)=-1000*x(1)-110*x(2)+cos(t)+cos(sqrt(1000)*t)+cos(1000*t);

  >> % key in the following instructions
  >> [t,x]=ode45(@second,[0:0.001:8],[0 0])
  >> for k=1:8001
        tk=0.001*(k-1);
        ylp(k)=1000*x(k,1);
        ybp(k)=110*x(k,2);
```

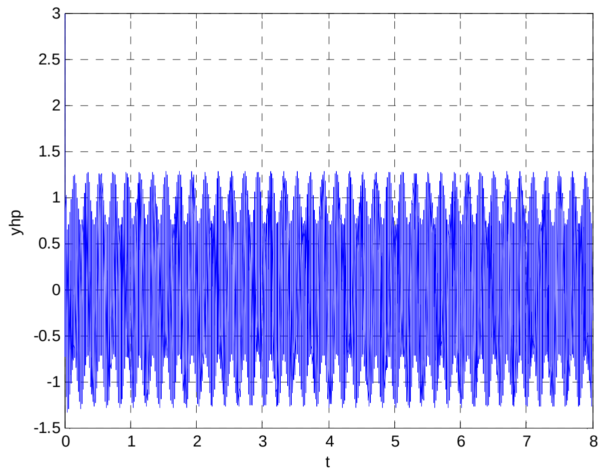
```
        u(k)=cos(tk)+ cos(sqrt(1000)*tk)+cos(1000*tk) ;
        yhp(k)=u(k)-ylp(k)-ybp(k) ;
    end
>> plot(t,ylp(:)); xlabel('t'); ylabel('ylp'); grid
```




```
>> plot(t,ybp(:)); xlabel('t'); ylabel('ybp'); grid
```




```
>> plot(t,yhp(:)); xlabel('t'); ylabel('yhp'); grid
```



==

P.1 Consider the following filter with transfer function as below:

$$H(s)=\frac{100s+1000}{s^2+100s+5000}$$

Draw the Bode plot and determine the frequency ω_1 at which the magnitude $|H(j\omega_1)|$ is maximal. If the input is $u(t)=cos(10\omega_1 t)$, what is the output $y(t)$?

10. Implementation of Second-Order Filters

In general, a filter can be implemented by electrical elements, such as resistors, capacitors and inductors. For example, an RLC circuit in Figure-1 is constructed to realize a low-pass filter where $v_s(t)$ is the input voltage and $v_L(t)$ is the output voltage to the payload resistance R_L. Its input-output equation is described as

$$\ddot{v}_L(t)+\left(\frac{R}{L}+\frac{1}{R_L L}\right)\dot{v}_L(t)+\frac{R_L+R}{R_L LC}v_L(t)=\frac{1}{LC}v_s(t) \qquad (1)$$

and the transfer function is

$$H(s)=\frac{V_L(s)}{V_s(s)}=k_0\frac{a_0}{s^2+a_1 s+a_0} \qquad (2)$$

where $a_1=\frac{R}{L}+\frac{1}{R_L L}$, $a_0=\frac{R_L+R}{R_L LC}$ and $k_0=\frac{R_L}{R_L+R}$. Although a low-pass filter can be easily implemented by an RLC circuit, there exists a serious problem concerning about the payload effect, i.e., the transfer function varies as the payload R_L changes. To avoid such undesired problem, a circuit including operational amplifiers, or Op-Amp in short, is often adopted as a substitute.

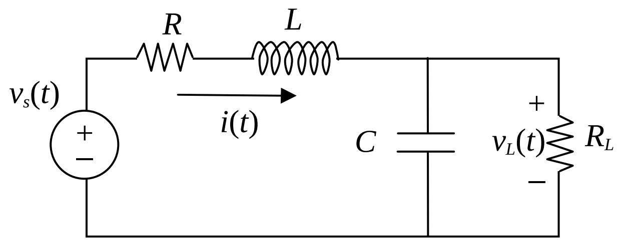


Figure-1

Let's employ Figure-2 as an example of a low-pass filter which is implemented by a circuit with an ideal Op-Amp. First, let's derive its transfer function. Since the current and voltage input to the Op-Amp are negligible, the node voltage $v_2(t)\approx 0$ is often considered to connect a virtual ground and both the currents passing through R_2 and C_2 are the same and denoted as $i(t)$. Therefore, we have

$$\frac{V_1(s)}{R_2} = -V_L(s)(sC_2) \tag{3}$$

where $V_1(s)$ and $V_L(s)$ are the Laplace transforms of $v_1(t)$ and $v_L(t)$. Moreover, based on the Kirchhoff's current law we obtain

$$\frac{V_1(s)-V_s(s)}{R_1} + V_1(s)(sC_1) + \frac{V_1(s)}{R_2} + \frac{V_1(s)-V_L(s)}{R_3} = 0 \tag{4}$$

where $V_s(s)$ are the Laplace transforms of $v_s(t)$. Substituting (3) into (4) leads to

$$\left(s^2R_2C_1C_2 + sR_2C_2\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)+\frac{1}{R_3}\right)V_L(s) = -\frac{V_s(s)}{R_1} \tag{5}$$

which implies the transfer function is

$$H(s) = \frac{V_L(s)}{V_s(s)} = k_0\,\frac{a_0}{s^2+a_1s+a_0} \tag{6}$$

where $a_1 = \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)$, $a_0 = \frac{1}{R_3R_2C_1C_2}$ and $k_0 = -\frac{R_3}{R_1}$. As a consequence, $H(s)$ is not affected by the payload resistance, but the gain k_0 is negative different from the case in (2). To change the numerical sign, an extra Op-Amp circuit is attached in Figure-3, which is then changed into a low-pass filter with a positive gain $k_0 = \frac{R_3}{R_1}$ in (6).

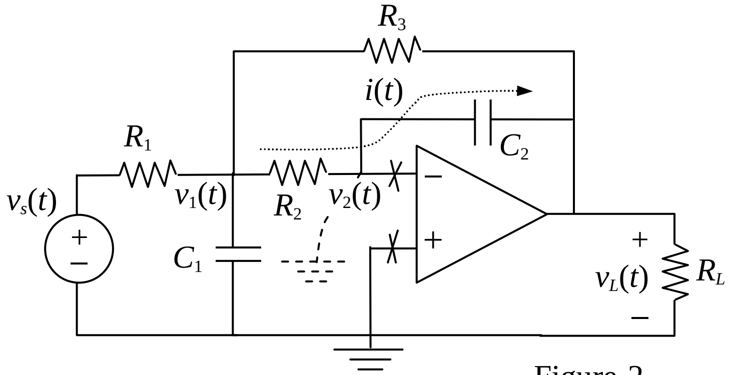


Figure-2

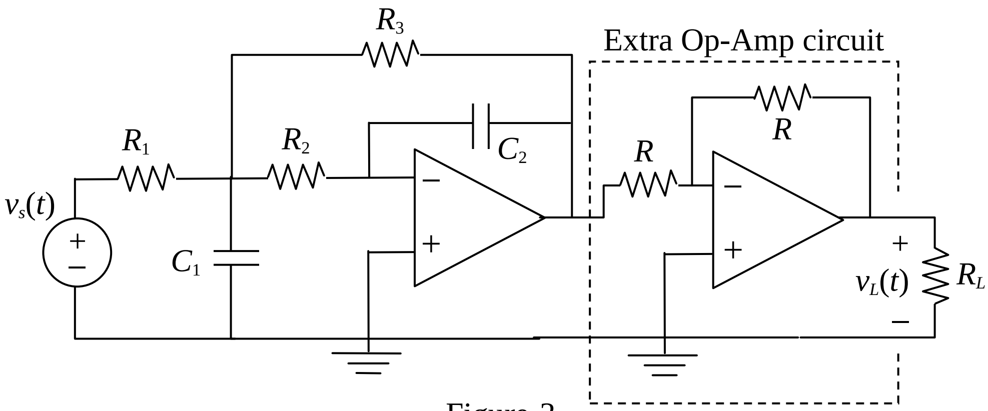


Figure-3

Next, let's first focus on the implementation of the 2nd order filter with transfer function expressed by

$$H(s) = \frac{1}{s^2 + a_1 s + a_0} \tag{7}$$

where $a_1>0$ and $a_0>0$. Mathematically, it can be realized by the following state equation

$$\dot{x}_1(t) = x_2(t) \tag{8}$$

$$\dot{x}_2(t) = -a_0 x_1(t) - a_1 x_2(t) \tag{9}$$

and the output equation is

$$y(t) = x_1(t) \tag{10}$$

From (8), we have

$$x_1(t) = \int_0^t x_2(\tau) d\tau \tag{11}$$

which implies an integrator is needed. In practice, the integrator is constructed by the Op-Amp circuit shown in Figure-4, described as

$$x_1(t) = -\int_0^t x_2(\tau) d\tau \tag{12}$$

which is different from (11). In order to simplify the implementation, we purposely change the state equation (8) and (9) into

$$\dot{x}_1(t) = -x_2(t) \tag{13}$$

$$-\dot{x}_2(t) = -a_0 x_1(t) + a_1 x_2(t) + u(t) \tag{14}$$

and take the same output $y(t) = x_1(t)$ in (10). Correspondingly, Figure-5 shows its block diagram.

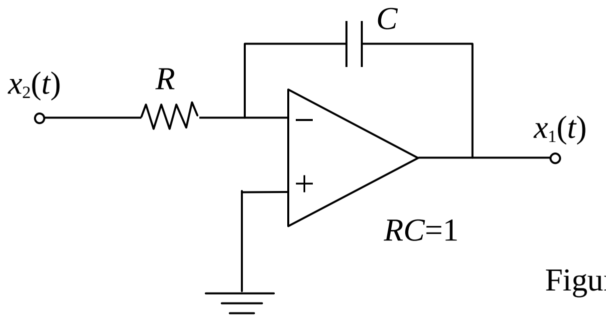


Figure-4

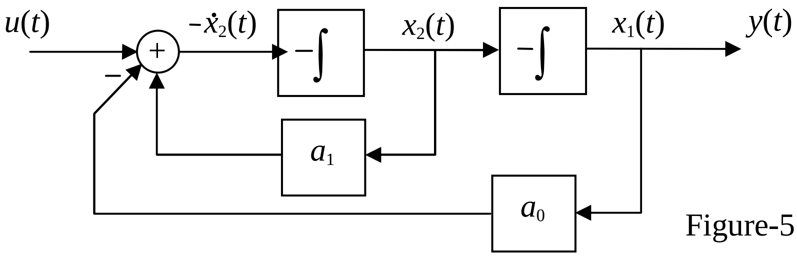


Figure-5

Viewing from Figure-5, it is clear that we still need to implement the inverter and adder, respectively shown in Figure-6 and Figure-7. Further rearranging (14) yields

$$-\dot{x}_2(t) = -[a_0 x_1(t) + (-[a_1 x_2(t) + u(t)])] \quad (15)$$

Figure-8 shows the block diagram correspondingly. Hence, the filter (7) can be implemented by the circuit with Op-Amps shown in Figure-9.

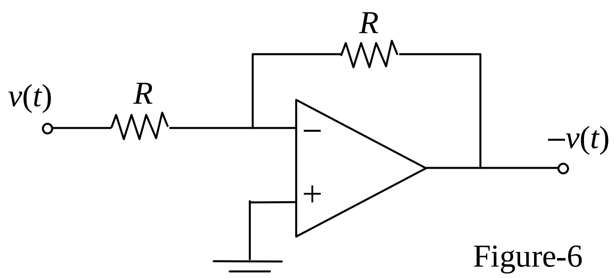


Figure-6

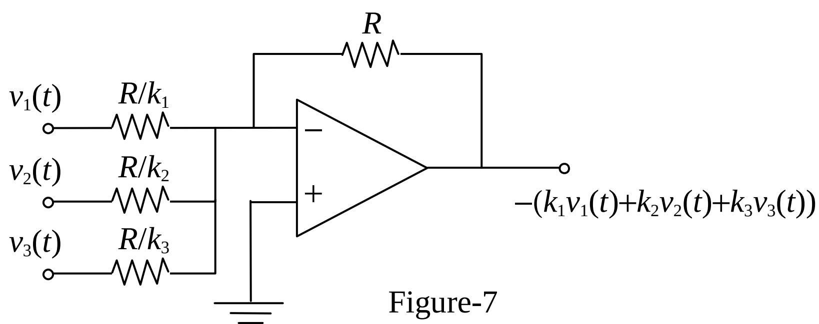


Figure-7

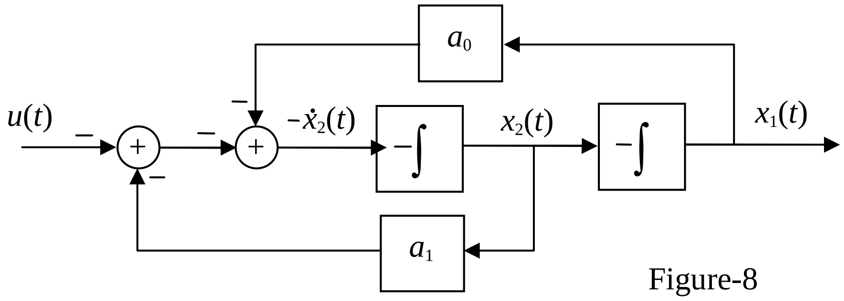


Figure-8

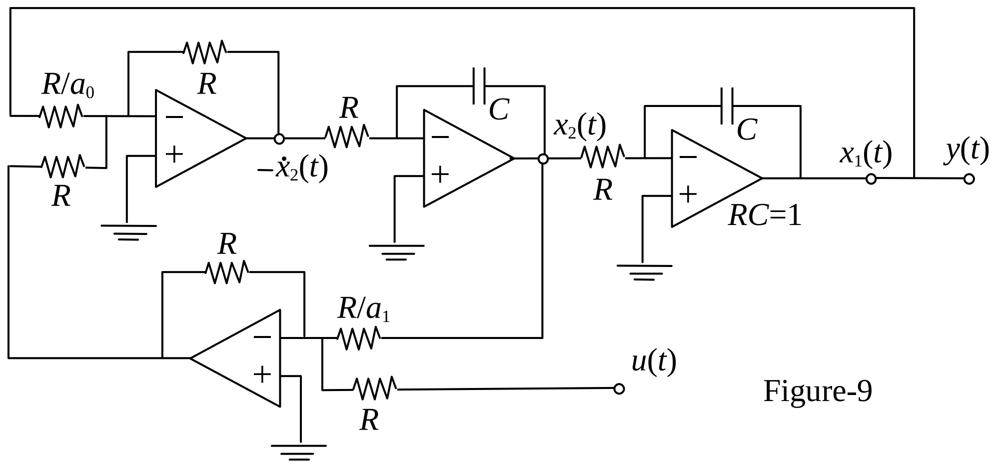


Figure-9

Based on Figure-9, we can implement the low-pass filter, band-pass filter and high-pass filter, respectively expressed as below:

$$H(s) = k_0 \frac{a_0}{s^2 + a_1 s + a_0} \tag{16}$$

$$H(s) = k_1 \frac{a_1 s}{s^2 + a_1 s + a_0} \tag{17}$$

$$H(s) = k_2 \frac{s^2}{s^2 + a_1 s + a_0} \tag{18}$$

Here, we will leave them as homework to draw their circuits with Op-Amps.

P.1 Implement the low-pass filter (16), band-pass filter (17) and high-pass filter (18) by the use of circuits with Op-Amps.