6. First-Order LTI Systems

A. First-Order LTI systems

B. First-Order Filter: RC Circuit

Linear time-invariant systems, or briefly called LTI systems, are the most important systems in engineering even though they are ideal, not real.

A. First-Order LTI Systems

The simplest dynamic system is a first-order LTI system shown in Figure 6-1. Mathematically, it is expressed by a state equation and an output equation, given as below:

$$\dot{x}(t) = ax(t) + bu(t), \qquad x(0)=x_0 \tag{6-1}$$

$$y(t) = cx(t) + du(t) \tag{6-2}$$

where $x(t)$ is the state variable representing the system intrinsic feature and $x(0)=x_0$ is its initial state at the initial time t=0. Besides, the system is excited by the external input $u(t)$ to generate the output response $y(t)$ accordingly.

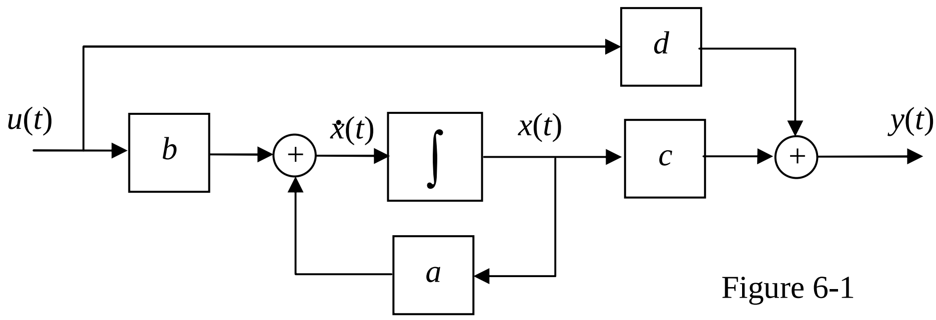

Figure 6-1

From the state equation (1), the system state $x(t)$ for $t\geq 0$, starting from the initial time t=0, can be obtained as

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \tag{6-3}$$

and then the output equals to

$$y(t) = ce^{at}x_0 + c\int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t) \tag{6-4}$$

which is measurable.

In addition to the model described by (1) and (2), an LTI system could be expressed by a differential equation without the state variable $x(t)$. First, let's calculate the following equation:

$$\begin{aligned}\dot{y}(t)-ay(t)&=c\dot{x}(t)+d\dot{u}(t)-a(cx(t)+du(t))\\&=c(\dot{x}(t)-ax(t))+d\dot{u}(t)-adu(t)\\&=cbu(t)+d\dot{u}(t)-adu(t)\\&=d\dot{u}(t)+(cb-ad)u(t)\end{aligned}\qquad(6\text{-}5)$$

where the term $\dot{x}(t)-ax(t)$ is replaced by $bu(t)$ according to (6-1). Clearly, the system in Figure 6-1 can be changed into a first-order differential equation, called the input-output equation and expressed by

$$\dot{y}(t)+a_0y(t)=b_1\dot{u}(t)+b_0u(t),\qquad y(0)=y_0\qquad(6\text{-}6)$$

where $a_0=-a$, $b_1=d$, $b_0=cb\text{-}ad$ and $y_0=cx_0+du(0)$. It is known that if $b_1=d\neq0$ then (6-6) is a proper system and if $b_1=d=0$ then it is a strictly proper system, given as

$$\dot{y}(t)+a_0y(t)=b_0u(t),\qquad y(0)=y_0\qquad(6\text{-}7)$$

Figure 6-2 shows the block diagram of (6-7) correspondingly.

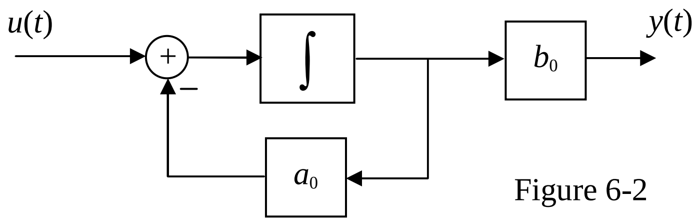

Figure 6-2

Obviously, the state equation and output equation of a first-order strictly proper LTI system is described as

$$\dot{x}(t)=ax(t)+bu(t),\qquad x(0)=x_0\qquad(6\text{-}8)$$

$$y(t)=cx(t)\qquad(6\text{-}9)$$

and the block diagram is shown in Figure 6-3. Since most of the systems in engneering are strictly proper, we will focus on this knid of systems from now on.

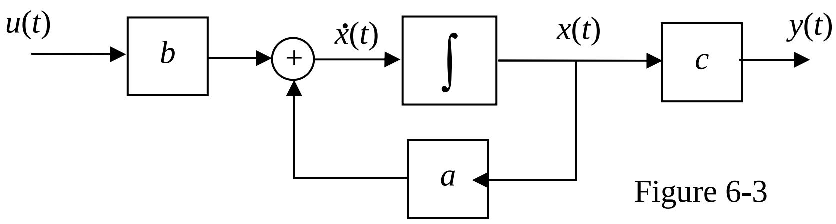

Figure 6-3

B. First-Order Filter: RC Circuit

In engineering, we often face problems related to signal processing, which usually requires a system, called filter, to extract desired signals from the input $u(t)$. Figure-4 shows the simplest low-pass filter implemented by an RC circuit, which only allows signals of low frequency to pass through.

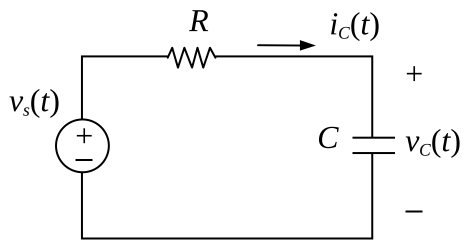

Figure 6-4

Now, let's derive the state equation by choosing the capacitor voltage $v_C(t)$ as the state variable due to its continuity in time domain. According to the Kirchhoff's voltage law, we have

$$R\,i_C(t)+v_C(t)=v_s(t), \qquad v_C(0)=v_{C0} \tag{6-10}$$

where $v_C(0)=v_{C0}$ is the initial voltage of the capacitor. The current through a capacitor is related to the voltage across it as below:

$$i_C(t)=C\dot{v}_C(t) \tag{6-11}$$

By substituting (6-11) into (6-10), we have

$$\dot{v}_C(t)=-\frac{1}{RC}v_C(t)+\frac{1}{RC}v_s(t), \qquad v_C(0)=v_{C0} \tag{6-12}$$

Let the state and input be $x(t)=v_C(t)$ and $u(t)=v_s(t)$, then (6-12) is rewritten as

$$\dot{x}(t)=ax(t)+bu(t), \qquad x(0)=x_0=v_{C0} \tag{6-13}$$

where $a=-\frac{1}{RC}$ and $b=\frac{1}{RC}$. To be used as a low-pass filter, the capacitor voltage is selected as the output, i.e.,

$$y(t)=v_C(t)=x(t) \tag{6-14}$$

which is equal to the state variable. Hence, from (6-13) we can write its input-output equation of the RC circuit as below:

$$\dot{y}(t)+a_0y(t)=b_0u(t), \qquad y(0)=y_0=v_{C0} \tag{6-15}$$

where $a_0=\frac{1}{RC}$ and $b_0=\frac{1}{RC}$. From (6-3), we have

$$\begin{aligned} y(t) = x(t) &= e^{-a_0 t} v_{C0} + \int_0^t e^{-a_0(t-\tau)} b_0 u(\tau) d\tau \\ &= e^{-\frac{1}{RC}t} v_{C0} + \frac{1}{RC}\int_0^t e^{-\frac{1}{RC}(t-\tau)} u(\tau) d\tau \end{aligned} \tag{6-16}$$

where the function $e^{-\frac{1}{RC}t}$ is shown in Figure 6-5 and T=RC is called the time constant. Obviously, if the time t increases by one time constant T, then the function decreases in a ratio of e^{-1}=0.3679. Moreover, the larger the time constant T is, the slower the function decreases. Note that $e^{-\frac{1}{RC}t} \rightarrow 0$ as $t \rightarrow \infty$. Therefore, when the time increases, the output (6-16) will be gradually approximated as below:

$$y(t) = x(t) \approx \frac{1}{RC}\int_0^t e^{-\frac{1}{RC}(t-\tau)} u(\tau) d\tau \tag{6-17}$$

which depends on the input $u(t)$ only. As a result, the initial condition y_0=v_{C0} is usually neglected in filter design. Next, let's check the frequency response of the RC circuit by the use of sinusoidal input $u(t)$=$cos\,\omega t$.

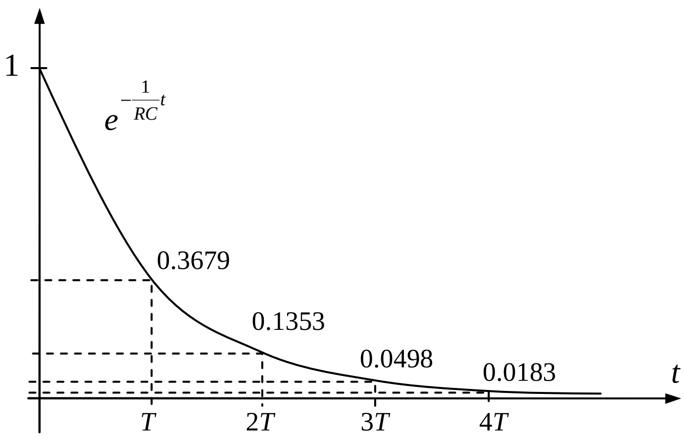

Figure 6-5

One important tool of frequency response is the Laplace transform. Let $U(s)$ and $Y(s)$ be the Laplace transform of $u(t)$ and $y(t)$, then (6-15) can be transformed as

$$sY(s) + a_0 Y(s) = b_0 U(s) \tag{6-18}$$

i.e.,

$$Y(s) = \frac{b_0}{s+a_0} U(s) = H(s)U(s) \tag{6-19}$$

where $H(s) = \frac{b_0}{s+a_0}$ is called the transfer function. If s is replaced by $j\omega$, then

$$H(s)\big|_{s=j\omega} = H(j\omega) = |H(j\omega)|e^{-j\angle H(j\omega)} \tag{6-20}$$

where the magnitude of $H(j\omega)$ is

$$|H(j\omega)| = \left|\frac{b_0}{j\omega + a_0}\right| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{1}{\sqrt{1+(\omega RC)^2}} \tag{6-21}$$

and the phase is

$$\angle H(j\omega) = -tan^{-1}(\omega RC) \tag{6-22}$$

It is well known that if the input is $u(t)=cos\,\omega t$, then its output is

$$y(t) = |H(j\omega)|cos(\omega t + \angle H(j\omega)) \tag{6-23}$$

which implies the output possesses the same frequency ω as the input signal, but its magnitude and phase are changed by $|H(j\omega)|$ and $\angle H(j\omega)$, shown in Figure 6-6. From (6-21), $|H(j\omega)|$ is decreased as the frequency ω is increased. Bisides, when $\omega = \omega_c = \dfrac{1}{RC}$, we have

$$|H(j\omega_c)| = \frac{1}{\sqrt{1+(\omega_c RC)^2}} = \frac{1}{\sqrt{2}} \tag{6-24}$$

$$\angle H(j\omega_c) = -tan^{-1}(\omega_c RC) = -45° \tag{6-25}$$

and then

$$y(t) = |H(j\omega_c)|cos(\omega t + \angle H(j\omega_c)) = \frac{1}{\sqrt{2}}cos(\omega t - 45°) \tag{6-26}$$

which implies the signal's power, in a form concerning $y^2(t)$, at $\omega = \omega_c$ is reduced to one half. Hence, $\omega = \omega_c$ is called the half power frequency or the cutoff frequency. To sum up, an input $u(t)=cos\,\omega t$ with lower frequency ω will generate an output with larger magnitude $|H(j\omega)|$. In other words, the filter $H(s) = \dfrac{b_0}{s+a_0}$ acts as a low-pass filter since it allows input signals of lower frequency to pass through easier.

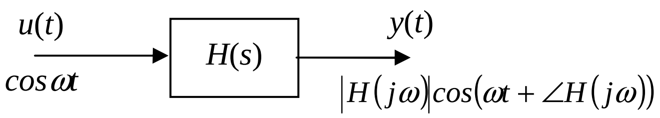

Figure 6-6

Let's take a numerical example to demonstrate the feature of the RC filter in

Figure 6-4, where R=1$k\Omega$, C=10μF and the initial capacitor voltage is 1V. From (6-12), we write the input-output equation as below:

$$\dot{v}_C(t) = -100v_C(t) + 100v_s(t), \qquad v_C(0)=1 \qquad (6\text{-}26)$$

What are the output responses for $u(t)=\cos\omega t$, where ω=1, 10, 100 and 100? The numerical answer is obtained by MATLAB and given as below:

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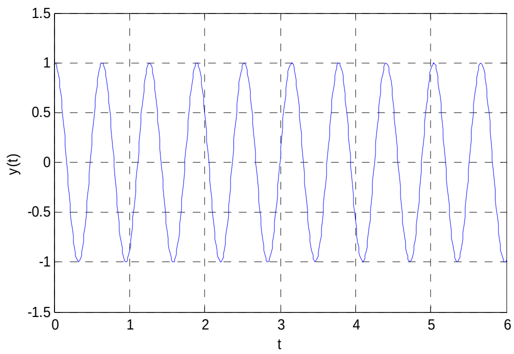
```
Create m-file: RC1.m
function dy=RC1(t,y)
dy=-100*y+100*cos(t);

Create m-file: RC2.m
function dy=RC2(t,y)
dy=-100*y+100*cos(10*t);

Create m-file: RC3.m
function dy=RC3(t,y)
dy=-100*y+100*cos(100*t);

Create m-file: RC4.m
function dy=RC4(t,y)
dy=-100*y+100*cos(1000*t);

>> % key in the following instructions
>> [t,y]=ode23(@RC1,[0:0.01:6],1.0)
>> plot(t,y); xlabel('t'); ylabel('y(t)')
```

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>> % key in the following instructions
>> [t,y]=ode23(@RC2,[0:0.01:6],1)
>> plot(t,y); xlabel('t'); ylabel('y(t)')
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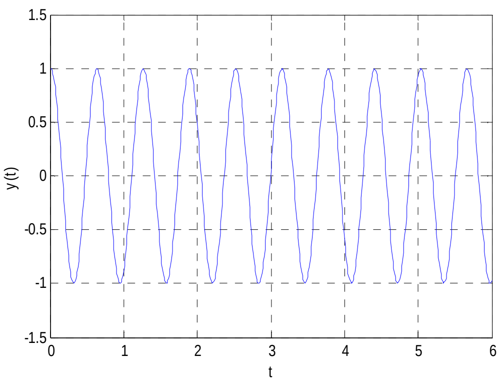


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>> % key in the following instructions
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>> [t,y]=ode23(@RC3,[0:0.01:6],1)
>> plot(t,y); xlabel('t'); ylabel('y(t)')
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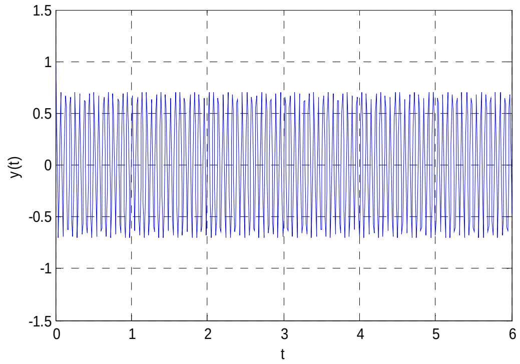


```
>> % key in the following instructions
>> [t,y]=ode23(@RC4,[0:0.01:6],1)
>> plot(t,y); xlabel('t'); ylabel('y(t)')
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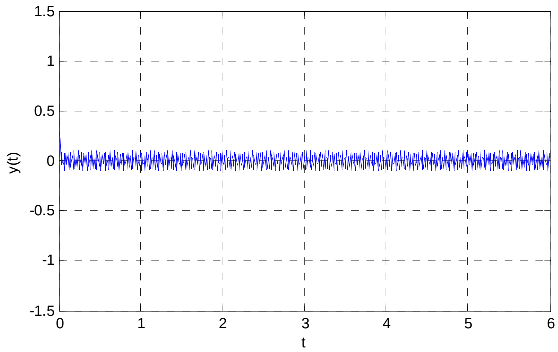

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The numerical result shows that the RC circuit indeed act as a low-pass filter since the magnitude of output is decreased as the input frequency ω is increased.

Problems

P.6-1 If the RC filter in Figure 6-4 contains R=16$k\Omega$, C=40μF and neglect the initial capacitor voltage. What is the cutoff frequency ω_c and what are the output responses for $u(t)=cos\,\omega t$ for ω=0.1ω_c, ω_c, 10ω_c and 100ω_c?

P.6-2 Replace the capacitor in Figure 6-4 by an inductor L. If R=16$k\Omega$, L=10mH and neglect the initial condition, then what is the cutoff frequency ω_c and what are the output responses for $u(t)=cos\,\omega t$ for ω=0.1ω_c, ω_c, 10ω_c and 100ω_c?