1. Static and Dynamic Systems

A. Static Systems
B. Dynamic Systems
C. Linear Time-Invariant Systems

In general, a practical system usually changes with time $t$ while in operating. As shown in Figure 1-1, a system receives external time-dependent input $u(t)$ and generates output $y(t)$ correspondingly. Based on the relation between $u(t)$ and $y(t)$, there are two types of systems, named as static systems and dynamic systems. For convenience, we denote the input and the output by $u(t)=[u_1(t), u_2(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ and $y(t)=[y_1(t), y_2(t), \ldots, y_p(t)]^T \in \mathbb{R}^p$, where the input contains $m$ input variables $u_i(t)$, $i=1,2,\ldots,m$, and the output contains $p$ output variables $y_j(t)$, $j=1,2,\ldots,p$.

\[ u(t) \xrightarrow{\text{System}} y(t) \]

Figure 1-1

A. Static Systems

A system is said to be static if its output $y(t)$ depends only on the input $u(t)$ at the present time $t$, mathematically described as

\[ y(t) = h(u(t)) = h(u_1(t), u_2(t), \ldots, u_m(t)) \]  \hspace{1cm} (1-1)

where $h(u(t)) = h(u_1(t), u_2(t), \ldots, u_m(t)) \in \mathbb{R}^p$ is a vector and its $j$-th component $h_j(u(t))$, $j=1,2,\ldots,p$, is an algebraic function of the input variables $u_1(t), u_2(t), \ldots, u_m(t)$. Figure 1-2 gives an example of static systems, which is a resistive circuit excited by an input voltage $u(t)$. Let the output be the voltage across the resistance $R_3$, and according to the circuit theory, we have
\[ y(t) = h(u(t)) = \frac{R_2R_3}{R_4(R_2 + R_3) + R_2R_3} u(t) = k \cdot u(t) \]  

(1-2)

where \( h(u(t)) \) is a function proportional to the present input \( u(t) \) and the constant gain is \( k = \frac{R_2R_3}{R_4(R_2 + R_3) + R_2R_3} \). Clearly, the output can be simply determined by the present input \( u(t) \), so it is a static system. In fact, a static system is also referred to as a memoryless system since its output response \( y(t) \) is not influenced by the past of input \( u(\tau) \) where \( \tau \leq t \).

\[ R_1 \]

\[ R_2 \]

\[ R_3 \]

\[ u(t) \]

\[ y(t) \]

\[ + \]

\[ - \]

\[ \text{Figure 1-2} \]

\section*{B. Dynamic Systems}

Unlike a static system, the output \( y(t) \) of a dynamic system will be affected by the input \( u(\tau) \) for \( \tau \leq t \), i.e., not only by the input \( u(t) \) at the present time but also its past \( u(\tau) \) for \( \tau \leq t \). Mathematically, a dynamic system can be expressed by the so-called state space description, which describes a system by its state equation and output equation. The state equation is often given as a first order differential equation:

\[ \dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0 \]  

(1-3)

and the output equation is in the form of

\[ y(t) = g(t, x(t), u(t)) \]  

(1-4)

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) is called the system state composed of \( n \) state variables \( x_i(t), i = 1, 2, \ldots, n \), and \( x(t_0) = x_0 \) is the system state at the initial time \( t = t_0 \).
usually called the initial state. Note that the system state represents the intrinsic features of the system. As for the output equation (1-4), it can be further changed into the following form

\[ y(t) = h(t, u(t)) \] (1-5)

due to the fact that the system state \( x(t) \) can be solved from (1-3) as a function of input \( u(t) \). Figure 1-3 shows the block diagram of the dynamic system expressed in state-space description, (1-3) and (1-4).

Now, one question is raised: Do we really know everything about the system just by the measured output \( y(t) \)? The answer is certainly “No!” since the output equation (1-4) only displays part of the system behavior. Actually, if you want to know the whole system behavior, you need to find all the state variables \( x_i(t) \), \( i = 1, 2, \ldots, n \), i.e., by solving the state equation (1-3). Unfortunately, this is not an easy work. Later, we will only focus on the most common and easiest case: linear time-invariant system.

**C. Linear Time-Invariant Systems**

The most important dynamic systems in engineering are called the linear time-invariant systems or LTI systems in brief. Its dynamic behavior is expressed by the following equations:

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \] (1-6)
\[ y(t) = Cx(t) + Du(t) \quad (1-7) \]

where all the system matrices \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \) are constant. It is well known that the system state \( x(t) \) can be solved from the state equation (1-6) as

\[ x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \quad (1-8) \]

Clearly, the system state \( x(t) \) is uniquely determined if the initial state \( x(t_0) \) and the input \( u(\tau) \) for \( t_0 \leq \tau \leq t \) are given. The output in (1-7) is then obtained as

\[ y(t) = C e^{A(t-t_0)} x(t_0) + C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \quad (1-9) \]

Since the output \( y(t) \) contains the integral of the input \( u(t) \), we know that the output \( y(t) \) is affected by the present input \( u(t) \) and all its past. That means the LTI system is indeed a dynamic system, not a static system.

**Problems**

P.1-1 Define \( e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k \). Show that if \( x(t) = e^{At} \), then \( \dot{x}(t) = Ae^{At} = e^{At}A \).

P.1-2 Given \( f(t) = \int_{t_0}^t g(t, \tau) d\tau \) where \( t_0 \) is a constant, show that the differentiation of \( f(t) \) is

\[ \frac{df(t)}{dt} = g(t, t) + \int_{t_0}^t \frac{\partial g(t, \tau)}{\partial t} d\tau \]

P.1-3 By direct calculation, verify that \( x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \) is the solution of \( \dot{x}(t) = Ax(t) + Bu(t) \) under the initial condition \( x(t_0) = x_0 \).