1. Static and Dynamic Systems

A. Static Systems
B. Dynamic Systems
C. Linear Time-Invariant Systems

In general, a practical system usually changes with time t while in operating. As shown in Figure 1-1, a system receives external time-dependent input $\boldsymbol{u}(t)$ and generates output $\boldsymbol{y}(t)$ correspondingly. Based on the relation between $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$, there are two types of systems, named as static systems and dynamic systems. For convenience, we denote the input and the output by $\boldsymbol{u}(t)=[u_1(t)\ u_2(t)\ \cdots\ u_m(t)]^T \in \Re^m$ and $\boldsymbol{y}(t)=[y_1(t)\ y_2(t)\ \cdots\ y_p(t)]^T \in \Re^p$, where the input contains m input variables $u_i(t)$, i=1,2,…,m, and the output contains p output variables $y_j(t)$, j=1,2,…,p.

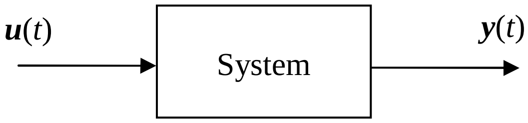

Figure1-1

A. *Static Systems*

A system is said to be static if its output $\boldsymbol{y}(t)$ depends only on the input $\boldsymbol{u}(t)$ at the present time t, mathematically described as

$$\boldsymbol{y}(t)=\boldsymbol{h}(\boldsymbol{u}(t))=\boldsymbol{h}(u_1(t),u_2(t),\cdots,u_m(t)) \qquad (1\text{-}1)$$

where $\boldsymbol{h}(\boldsymbol{u}(t))=\boldsymbol{h}(u_1(t),u_2(t),\cdots,u_m(t))\in\Re^p$ is a vector and its j-th component $h_j(\boldsymbol{u}(t))$, j=1,2,…,p, is an algebraic function of the input variables $u_1(t)$, $u_2(t)$, …, $u_m(t)$. Figure 1-2 gives an example of static systems, which is a resistive circuit excited by an input voltage $u(t)$. Let the output be the voltage across the resistance R_3, and according to the circuit theory, we have

$$y(t) = h(u(t)) = \frac{R_2 R_3}{R_1(R_2 + R_3) + R_2 R_3} u(t) = k \cdot u(t) \qquad (1\text{-}2)$$

where $h(u(t))$ is a function proportional to the present input $u(t)$ and the constant gain is $k = \frac{R_2 R_3}{R_1(R_2 + R_3) + R_2 R_3}$. Clearly, the output can be simply determined by the present input $u(t)$, so it is a static system. In fact, a static system is also referred to as a memoryless system since its output response $y(t)$ is not influenced by the past of input $u(\tau)$ where $\tau < t$.

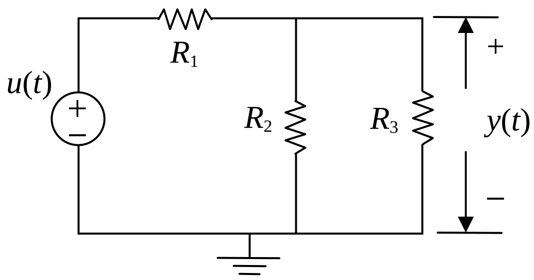

Figure 1-2

B. Dynamic Systems

Unlike a static system, the output $\boldsymbol{y}(t)$ of a dynamic system will be affected by the input $\boldsymbol{u}(\tau)$ for $\tau \le t$, i.e., not only by the input $\boldsymbol{u}(t)$ at the present time but also its past $\boldsymbol{u}(\tau)$ for $\tau < t$. Mathematically, a dynamic system can be expressed by the so-called state space description, which describes a system by its state equation and output equation. The state equation is often given as a first order differential equation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \qquad (1\text{-}3)$$

and the output equation is in the form of

$$\boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \qquad (1\text{-}4)$$

where $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \Re^n$ is called the system state composed of n state variables $x_i(t)$, $i=1,2,\ldots,n$, and $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is the system state at the initial time $t = t_0$, usually called the initial state. Note that the system state represents the intrinsic features of the system. As for the output equation (1-4), it can be further changed into the following form

$$\boldsymbol{y}(t) = \boldsymbol{h}(t, \boldsymbol{u}(t)) \qquad (1\text{-}5)$$

due to the fact that the system state $\boldsymbol{x}$(t) can be solved from (1-3) as a function of input $\boldsymbol{u}$(t). Figure 1-3 shows the block diagram of the dynamic system expressed in state-space description, (1-3) and (1-4).

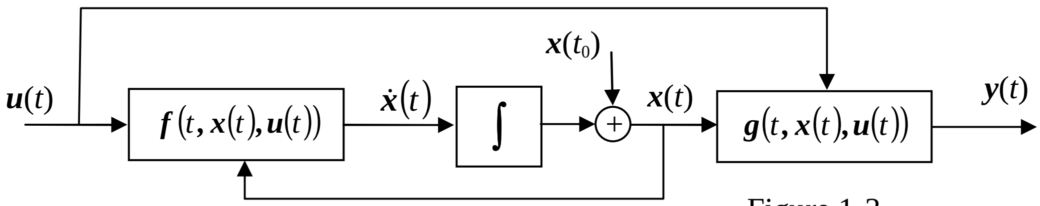

Figure 1-3

Now, one question is raised: Do we really know everything about the system just by the measured output $\boldsymbol{y}(t)$? The answer is certainly “No!” since the output equation (1-4) only displays part of the system behavior. Actually, if you want to know the whole system behavior, you need to find all the state variables $x_i(t)$, i=1,2,…,n, i.e., by solving the state equation (1-3). Unfortunately, this is not an easy work. Later, we will only focus on the most common and easiest case: linear time-invariant system.

C. Linear Time-Invariant Systems

The most important dynamic systems in engineering are called the linear time-invariant systems or LTI systems in brief. Its dynamic behavior is expressed by the following equations:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \qquad (1\text{-}6)$$

$$\boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t) \tag{1-7}$$

where all the system matrices $\boldsymbol{A}\in\Re^{n\times n}$, $\boldsymbol{B}\in\Re^{n\times m}$, $\boldsymbol{C}\in\Re^{p\times n}$ and $\boldsymbol{D}\in\Re^{p\times m}$ are constant. It is well known that the system state $\boldsymbol{x}(t)$ can be solved from the state equation (1-6) as

$$\boldsymbol{x}(t)=e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0+\int_{t_0}^{t}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau \tag{1-8}$$

Clearly, the system state $\boldsymbol{x}(t)$ is uniquely determined if the initial state $\boldsymbol{x}(t_0)$ and the input $\boldsymbol{u}(\tau)$ for $t_0\le\tau\le t$ are given. The output in (1-7) is then obtained as

$$\boldsymbol{y}(t)=\boldsymbol{C}\,e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0)+\boldsymbol{C}\int_{t_0}^{t}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau+\boldsymbol{D}\boldsymbol{u}(t) \tag{1-9}$$

Since the output $\boldsymbol{y}(t)$ contains the integral of the input $\boldsymbol{u}(t)$, we know that the output $\boldsymbol{y}(t)$ is affected by the present input $\boldsymbol{u}(t)$ and all its past. That means the LTI system is indeed a dynamic system, not a static system.

Problems

P.1-1 Define $e^{\boldsymbol{A}t}=\sum_{k=0}^{\infty}\frac{1}{k!}(\boldsymbol{A}t)^k$. Show that if $\boldsymbol{x}(t)=e^{\boldsymbol{A}t}$, then $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}e^{\boldsymbol{A}t}=e^{\boldsymbol{A}t}\boldsymbol{A}$.

P.1-2 Given $f(t)=\int_{t_0}^{t}g(t,\tau)d\tau$ where t_0 is a constant, show that the differentiation of $f(t)$ is $\frac{df(t)}{dt}=g(t,t)+\int_{t_0}^{t}\frac{\partial g(t,\tau)}{\partial t}d\tau$

P.1-3 By direct calculation, verify that $\boldsymbol{x}(t)=e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0+\int_{t_0}^{t}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$ is the solution of $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t)$ under the initial condition $\boldsymbol{x}(t_0)=\boldsymbol{x}_0$.