11. Mass-Damper-Spring Systems

In engineering, most of the systems are constructed by mechanical components such as mass, damper and spring. The simplest mechanical system is called the mass-damper-spring system, or MBK system in brief, which is depicted in Figure-1 where M is the mass of the moving object, B is the damping coefficient of the damper and K is the stiffiness of the spring. Let $f(t)$ be an extra force exerted on the object and assume $x_M(t)$ is the resulted deviation of the spring referred to its unforced status $x_{M0}=0$. Then, there are two forces reacted to restrain the motion of the object, expressed as

$$f_B(t) = -B\dot{x}_M(t) \tag{1}$$

$$f_K(t) = -Kx_M(t) \tag{2}$$

where $f_B(t)$ is caused by the damper and $f_K(t)$ is the spring force. According to the Newton's Second Law of Motion, we have

$$f(t) + f_B(t) + f_K(t) = M\ddot{x}_M(t) \tag{3}$$

i.e.,

$$M\ddot{x}_M(t) + B\dot{x}_M(t) + Kx_M(t) = f(t) \tag{4}$$

which is the dynamic equation of the MBK system.

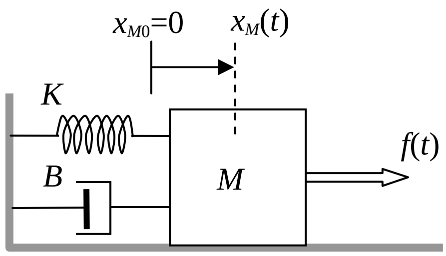

Figure-1

For the mechanical system, the external force $f(t)$ is the input and the deviation $x_M(t)$ is commonly chosen as the output. Let the Laplace transforms be $F(s)$ and $X_M(s)$, then the system (4) can be described as

$$\left(Ms^2+Bs+K\right)X_M(s)=F(s) \tag{5}$$

The transfer function is then obtained as

$$H(s)=\frac{X_M(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}=k_0\,\frac{a_0}{s^2+a_1s+a_0} \tag{6}$$

where $a_1=\dfrac{B}{M}$, $a_0=\dfrac{K}{M}$ and $k_0=\dfrac{1}{K}$. Clearly, it is a low-pass filter and can reject high-frequency inputs. For convenience, we define $a_0=\omega_n^2$ and $a_1=2\xi\omega_n$ and rewrite the transfer function in (6) as

$$H(s)=k_0\,\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \tag{7}$$

where ω_n is the natural frequency and ξ is the damping ratio.

Actually, the MBK system (7) can be classified into two types referring to the damping ratio ξ. For the first type, if $\xi\geq 1$ then the roots of $s^2+2\xi\omega_n s+\omega_n^2=0$ are real numbers, which are also called the poles of the system. Denote the poles as s_1 and s_2 and thus,

$$s_1, s_2=\left(-\xi\pm\sqrt{\xi^2-1}\right)\omega_n \tag{8}$$

In this case, the magnitude of the output is decreased monotonically when the frequency of the sinusoidal input is increased. For the second type, if $\xi<1$ then the poles s_1 and s_2 are complex numbers in conjugate, expressed as

$$s_1, s_2=-\xi\omega_n\pm j\sqrt{1-\xi^2}\cdot\omega_n=-\xi\omega_n\pm j\omega_d \tag{9}$$

where $\omega_d=\sqrt{1-\xi^2}\cdot\omega_n$ is called the damped natural frequency. Unlike the first type, the magnitude of the output is increased when the frequency of the sinusoidal input is near to ω_n. Besides, if $\omega=\omega_n$, then the magnitude

$$\left|H(j\omega_n)\right|=\left|k_0\,\frac{\omega_n^2}{(j\omega_n)^2+2\xi\omega_n(j\omega_n)+\omega_n^2}\right|=\frac{k_0}{2\xi} \tag{9}$$

An example with k_0=0.1, ξ=0.01 and ω_n=100 is given as below:

$$H(s) = \frac{1000}{s^2 + 2s + 10000} \tag{10}$$

Its Bode plot is obtained as below:

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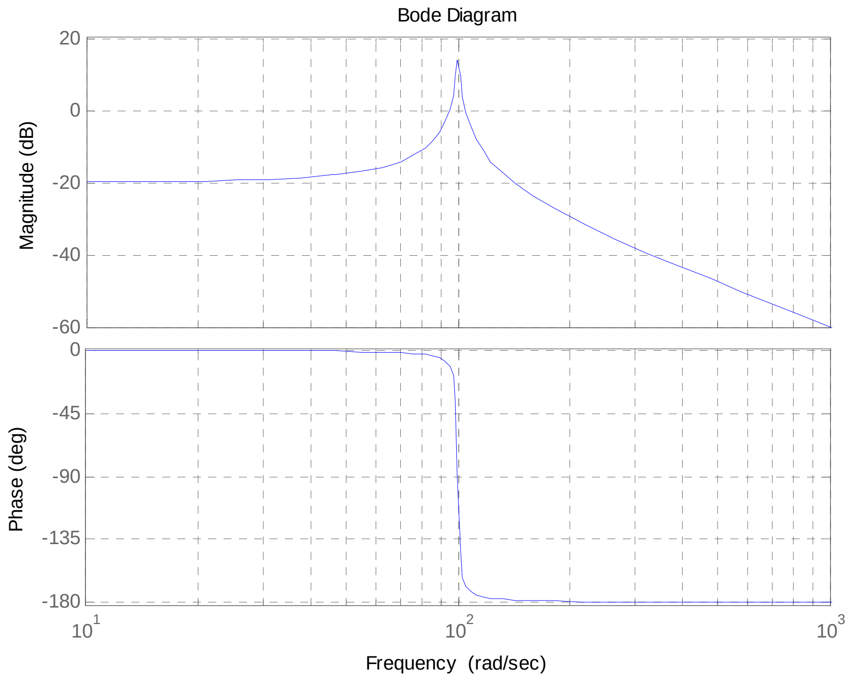
```
>>% Create transfer function
>>numlp=[0 0 1000]; den=[1 2 10000];
>>lpf=tf(numlp,den)

Transfer function:
        1000
----------------------------
s^2 + 2s + 10000

>>bode(lpf);grid
```



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It is clear that the maximum magnitude happens near to ω_n=100. Furthermore, let the input be a signal as below:

$$u(t) = \begin{cases} cos(50t) & \text{for } 0 < t \le 2 \\ cos(100t) & \text{for } 2 < t \le 2.5 \\ cos(150t) & \text{for } t > 2.5 \end{cases} \tag{11}$$

then the output can be obtained by the use of MATLAB.

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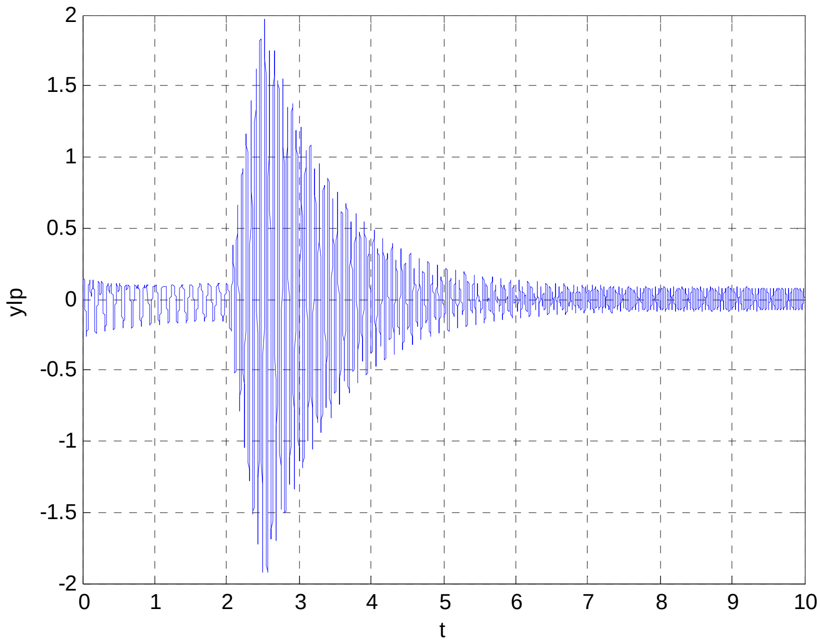
```
Create m-file: second.m
  function dx=second(t,x)
  dx=zeros(2,1);
  dx(1)=x(2);
  if t<=2
  dx(2)=-10000*x(1)-2*x(2)+cos(50*t);
  elseif t<=2.5
  dx(2)=-10000*x(1)-2*x(2)+cos(100*t);
  else
  dx(2)=-10000*x(1)-2*x(2)+cos(150*t);
  end

  >> % key in the following instructions
  >> [t,x]=ode45(@second,[0:0.001:10],[0 0]);
  >> for k=1:10001
        ylp(k)=1000*x(k,1);
     end
  >> plot(t,ylp(:)); xlabel('t'); ylabel('ylp'); grid
```



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It is obvious that the system is a little affected by the input for $0<t<2$ sec since the frequency ω=10 is much less than the natural frequency ω=100. However, for $t>2$ sec, the output become sharply increased due to the use of sinusoidal input of natural frequency. After $t<2.5$ sec, by taking away the natural frequency the output starts to decease, as expected.

P.1 Consider an MBK mechanical system with transfer function as below:

$$H(s)=\frac{1000}{s^2+s+2500}$$

Draw the Bode plot and determine the frequency ω_0 at which the magnitude $\left|H(j\omega_0)\right|$ is maximal. With the use of MATLAB, please find the output $y(t)$ when the input is $u(t)=cos((0.1\omega_0+0.2\omega_0 t)t)$ for $0<t<10$.

12. PID Control of the simplest Second Order System

In the field of filter design, we are asked to construct a system that can reject or extract part of the input signals with specific frequencies. Unlike the filter design, a different field called controller design is also important in system engineering, which is applied to systems, sometimes called plants, that could not operate properly as desired. To adjust the behaviour of a plant, we often build up a subsystem known as the controller connected to the plant such that the controlled system can fulfill the desired function. Here, we will focus on the PID controller design of second order systems.

The PID control has been widely used in engineering because of its simplicity and robustness, especially for unknown plants or plants with uncertainties. Let's consider the following simplest second-order plant

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=u(t) \tag{1}$$

which may be a stable, unstable or oscillatory system if $u(t)=0$. To adjust the system's behavior, the PID control is set to be

$$u(t)=k_Pe(t)+k_I\int_{t_0}^{t}e(\tau)d\tau+k_D\dot{e}(t) \tag{2}$$

where k_P, k_I and k_D are coefficients respectively referring to three different types, called the proportional, integral and derivative, of the error signal $e(t)$. In general, the error signal is chosen as

$$e(t) = y(t) - y_d(t) \tag{3}$$

where $y_d(t)$ is the desired output. There are two kinds of control problems, named as tracking control and set-point regulation. When the desired output $y_d(t)$ is changed with time, the PID control is designed to track the time-varying desired output. On the other hand, if the desired output $y_d(t)= y_d$ is a fixed set-point, then the system is regulated by the PID control to the set-point.

First, let's discuss the set-point regulation, i.e., $y_d(t)=y_d$. It is easy to rewrite the system (1) as

$$\ddot{e}(t) + a_1\dot{e}(t) + a_0 e(t) + a_0 y_d = u(t) \tag{4}$$

and then the use of PID control (2) results in

$$\ddot{e}(t) + (a_1 - k_D)\dot{e}(t) + (a_0 - k_P)e(t) - k_I\int_{t_0}^{t} e(\tau)d\tau + a_0 y_d = 0 \tag{5}$$

Further differentiating (5) yields

$$\dddot{e}(t) + \alpha_2\ddot{e}(t) + \alpha_1\dot{e}(t) + \alpha_0 e(t) = 0 \tag{6}$$

where $\alpha_2 = a_1 - k_D$, $\alpha_1 = a_0 - k_P$ and $\alpha_0 = -k_I$. According to the Ruth-Hurwitz criterion, we know that under the conditions

$$\alpha_2 > 0 \text{ and } \alpha_1\alpha_2 > \alpha_0 > 0 \tag{7}$$

the system can be stabilized, otherwise it is unstable.

In case that the system is exactly known, i.e., the coefficients a_0 and a_1 are given, then we can pre-assign three distinct stable eigenvalues λ_1, λ_2 and λ_3 for the characteristic equation

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0 \tag{8}$$

which is derived from (6). Therefore,

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \tag{9}$$

It implies

$$\alpha_2 = a_1 - k_D = -(\lambda_1 + \lambda_2 + \lambda_3) \tag{10}$$

$$\alpha_1 = a_0 - k_P = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \tag{11}$$

$$\alpha_0 = -k_I = -\lambda_1\lambda_2\lambda_3 \tag{12}$$

Clearly, the PID controller in (2) is possessed of coefficients as below:

$$k_D = a_1 + \lambda_1 + \lambda_2 + \lambda_3 \tag{13}$$

$$k_P = a_0 - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1\lambda_3 \tag{14}$$

$$k_I = \lambda_1\lambda_2\lambda_3 \tag{15}$$

The use of these coefficients leads to the error signal expressed as

$$e(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} \tag{16}$$

where A_1, A_2 and A_3 are constant and determined by the system's initial condition $y(t_0)$ and $\dot{y}(t_0)$. Since the real parts of the eigenvalues are negative, we have

$$\lim_{t\to\infty} e(t) = 0 \tag{17}$$

which means the system is successfully controlled to the desired set-point $y_d(t)= y_d$.

What will happen if the integral part is not included in (2)? This kind of control is referred to as the PD control and chosen as

$$u(t) = k_P e(t) + k_D \dot{e}(t) \tag{18}$$

From (5) by neglecting k_I, we have

$$\ddot{e}(t) + \beta_1\dot{e}(t) + \beta_0 e(t) + a_0 y_d = 0 \tag{19}$$

where $\beta_1 = a_1 - k_D$ and $\beta_0 = a_0 - k_P$. The characteristic euquation is

$$\lambda^2 + \beta_1\lambda + \beta_0 = 0 \tag{20}$$

According to the Ruth-Hurwitz criterion, it is known that under the conditions

$$\beta_1 > 0 \quad \text{and} \quad \beta_0 > 0 \tag{21}$$

the system can be stabilized, otherwise it is unstable. Choose two distinct stable eigenvalues $\lambda_1$and λ_2 for the characteristic equation (20) and thus the error signal is obtained as

$$e(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} - \frac{a_0}{\beta_0} y_d \tag{22}$$

where B_1 and B_2 are constant and determined by the system's initial conditions $y(t_0)$ and $\dot{y}(t_0)$. Since the real parts of the eigenvalues are negative, we have

$$\lim_{t\to\infty} e(t) = -\frac{a_0}{\beta_0} y_d \neq 0 \tag{23}$$

which means the system is not successfully driven to the desired set-point $y_d(t)=y_d$. There indubitably exists a steady-state error if the integral part of control is not adopted.

Next, let's discuss the tracking control, i.e., $y_d(t)$ is time-dependent. Rewrite the system (1) as

$$\ddot{e}(t) + a_1\dot{e}(t) + a_0 e(t) + \ddot{y}_d(t) + a_1\dot{y}_d(t) + a_0 y_d(t) = u(t) \tag{24}$$

and then the use of PID control (2) results in

$$\begin{aligned}\ddot{e}(t) + (a_1 - k_D)\dot{e}(t) + (a_0 - k_P)e(t) - k_I \int_{t_0}^{t} e(\tau)d\tau \\ + \ddot{y}_d(t) + a_1\dot{y}_d(t) + a_0 y_d(t) = 0\end{aligned} \tag{25}$$

Further differentiating (25) yields

$$\dddot{e}(t) + \alpha_2\ddot{e}(t) + \alpha_1\dot{e}(t) + \alpha_0 e(t) + \dddot{y}_d(t) + a_1\ddot{y}_d(t) + a_0\dot{y}_d(t) = 0 \tag{26}$$

where $\alpha_2 = a_1 - k_D$, $\alpha_1 = a_0 - k_P$ and $\alpha_0 = -k_I$. According to the Ruth-Hurwitz criterion, we know that under the conditions

$$\alpha_2 > 0 \quad \text{and} \quad \alpha_1\alpha_2 > \alpha_0 > 0 \tag{27}$$

the system can be stabilized. Similarly, by choosing three distinct stable eigenvalues λ_1, λ_2 and λ_3 for the system, the error signal can be obtained as

$$e(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + g(t) \tag{28}$$

where A_1, A_2 and A_3 are constant and $g(t)$ is related to the desired output. Clearly, there exists a tracking error under PID control.

P.1 Consider an MBK mechanical system pulled by a fixed payload m_L and a control force $f(t)$ as shown in Figure-1, where the mass M=1 kg, the damping B=0.001 N·s/m, the stiffness K=100 N/m and the payload m_L=0.4 kg. Assume the constant of the gravity is g=9.8 m/s^2. Let the deflection of the spring be $x_M(t)$.

(A) Choose $x_M(t)$ as the output, then, what is the dynamic equation of the MBK system in the form of second order differential equation?

(B) If $f(t)$=0 and the initial conditions are $x_M(0)$=0 m and $\dot{x}_M(0) = 0\ m/s$, then what is $x_M(t)$ for t>0? Plot $x_M(t)$ for 0<t<10 sec in Matlab.

(C) It is easy to find from (B) that the MBK system will oscillate without control. Please design a PID control to make the mass move to $x_M(t)$=0 quickly and without oscillation.

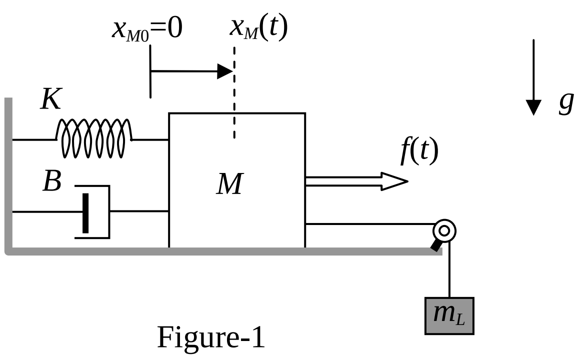

Figure-1